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With the Co-operation of SIMON NEWCOMB

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Eme Picon

Sur une transformation de mouvements.

PAR PAUL APPELL.

Un problème traité par M. Elliot (Comptes Rendus 1893 et Annales de l'Ecole Normale 1893) et une question résolue par M. Mestschersky (Bulletin des Sciences mathématiques 1894) peuvent être envisagés comme des cas particuliers d'une même transformation de mouvements.

1). Soient

$$\frac{d^2x_1}{dt^2} = X_1, \quad \frac{d^2x_2}{dt^2} = X_2, \quad \frac{d^2x_3}{dt^2} = X_3, \ldots . \qquad (1)$$

les équations du mouvement d'un ou plusieurs points libres, X_1, X_2, X_3, ... étant des fonctions de x_1, x_2, x_3, ... et du temps t.

Désignons par λ et μ deux fonctions d'une variable τ et faisons

$$x_1 = \lambda y_1, \quad x_2 = \lambda y_2, \quad x_3 = \lambda y_3, \ldots .$$
$$d\tau = \mu dt.$$

Appelons enfin λ', μ', λ'', les derivées successives de λ et μ par rapport à τ. Les équations deviennent

$$\frac{d^2y_1}{d\tau^2} + \left(\frac{2\lambda'}{\lambda} + \frac{\mu'}{\mu}\right)\frac{dy_1}{d\tau} + \left(\frac{\lambda''}{\lambda} + \frac{\lambda'\mu'}{\lambda\mu}\right) y_1 = Y_1, \qquad (2)$$

....

où on a posé

$$Y_1 = \frac{X_1}{\lambda\mu^2}, \ldots . \qquad (3)$$

Les quantités Y_1, Y_2, ... sont alors des fonctions de y_1, y_2, y_3, ... et τ.

En déterminant λ et μ de façon que l'on ait

$$\frac{2\lambda'}{\lambda} + \frac{\mu'}{\mu} = 2a, \quad \frac{\lambda''}{\lambda} + \frac{\lambda'\mu'}{\lambda\mu} = b, \qquad (4)$$

où a et b désignent des constantes, on voit que l'on peut déduire du mouvement (1) un second mouvement (2) dans lequel, à la force de projections $Y_1, Y_2, Y_3, \ldots,$ viennent s'ajouter une résistance $2a\frac{dy_1}{dt}, 2a\frac{dy_2}{dt}, \ldots.$ proportionnelle à la vitesse et une attraction d'un point fixe $by_1, by_2, \ldots.$ proportionnelle à la distance.

La transformation inverse permettra de passer du mouvement (2) au mouvement (1).

La première des équations (4) donne immédiatement

$$\lambda^2\mu = Ae^{2a\tau}, \tag{5}$$

A étant une constante arbitraire. La seconde devient, par l'élimination de μ,

$$\frac{\lambda''}{\lambda} - \frac{2\lambda'^2}{\lambda^2} + 2a\frac{\lambda'}{\lambda} - b = 0:$$

en y faisant $\frac{\lambda'}{\lambda} = \rho + a$ et posant $b - a^2 = c^2$, on a une équation en ρ

$$\rho' = \rho^2 + c^2,$$

qui admet pour intégrale générale

$$\rho = c \operatorname{tg}(c\tau + \theta)$$

(θ constante arbitraire) et pour intégrales singulieres

$$\rho = \pm ic.$$

On a donc pour λ

$$\lambda = \frac{Be^{a\tau}}{\cos(c\tau + \theta)}, \tag{6}$$

ou

$$\lambda = Be^{(a \pm ic)\tau}, \tag{7}$$

B désignant une constante. Dans les équations a et b sont des constantes réelles, c est une constante réelle ou purement imaginaire, de sorte que la solution (7) peut-être réelle. Appliquons ces formules à quelques cas particuliers.

2). *Problème de M. Elliot.* Soit

$$U(x_1, x_2, x_3 \ldots . t)$$

une fonction des coordonnées et du temps: supposons

$$X_1 = \frac{\partial U}{\partial x_1}, \; X_2 = \frac{\partial U}{\partial x_2}, \ldots.$$

Supposons en outre $b = 0$. Alors $e = \pm ia$. On a, dans ce cas, en particularisant les constantes, la solution simple suivante des équations (4):

$$\lambda = 1, \quad \mu = e^{2a\tau}.$$

Avec ce choix de λ et μ, on a

$$Y_1 = \frac{X_1}{\mu^2} = e^{-4a\tau} \frac{\partial U}{\partial x_1}, \ldots .$$

$$t = -\frac{1}{2a} e^{-2a\tau}.$$

Comme $x_1 = y_1$, $x_2 = y_2$, ..., on a, en posant,

$$V = e^{-4a\tau} U\left(y_1, y_2, y_3, \ldots . - \frac{1}{2a} e^{-2a\tau}\right)$$

les nouvelles équations

$$\frac{d^2 y_1}{d\tau^2} + 2a \frac{dy_1}{d\tau} = \frac{\partial V}{\partial y_1}, \ldots . \tag{8}$$

Par le changement de variable, on a donc, sans changer la forme des seconds membres des équations, fait apparaître une résistance de milieu. La transformation inverse la fera disparaître et ramènera les équations à la forme primitive

$$\frac{d^2 x_1}{dt^2} = \frac{\partial U}{\partial x_1}, \ldots . \tag{9}$$

à laquelle on peut appliquer les théorêmes de Jacobi. On pourra donc ramèner l'intégration des équations (8) à la méthode de Jacobi. C'est là le théorême que M. Elliot a démontré directement et dont il a fait plusieurs applications intéressantes.

3). *Problème de M. Mestschersky.* Supposons

$$X_1, X_2, X_3, \ldots .$$

indépendants de t et homogènes de degré -3 en $x_1, x_2, x_3, \ldots .$ Quand on remplace $x_1, x_2, x_3, \ldots .$ par $\lambda y_1, \lambda y_2, \lambda y_3, \ldots .$, la fonction

$$X_1 = \phi_1(x_1, x_2, x_3, \ldots .)$$

est remplacée par $\lambda^{-3}\phi_1(y_1, y_2, y_3, \ldots .)$.

Donc

$$Y_1 = \frac{X_1}{\lambda\mu^2} = \frac{\phi_1(y_1, y_2, y_3, \ldots .)}{\lambda^4\mu^2}.$$

Particularisons encore les calculs précédents en supposant $a = 0$. Alors on peut prendre, d'après (5) et (6),

$$\lambda^2\mu = 1, \quad \lambda = \frac{1}{\cos c\tau}$$

$$Y_1 = \phi_1(y_1, y_2, y_3, \ldots .).$$

Les équations du mouvement conserveront dans ce cas la même forme sauf l'adjonction des termes by_1, by_2, ... représentant une attraction proportionnelle à la distance. Si donc on sait trouver le mouvement défini par les équations

$$\frac{d^2x_k}{dt^2} = \phi_k(x_1, x_2, x_3, \ldots .),$$

où ϕ_k est homogène et de degré -3, on sait, par là même, trouver le mouvement défini par les équations

$$\frac{d^2y_k}{d\tau^2} + by_k = \phi_k(y_1, y_2, y_3, \ldots .).$$

C'est là le théorême de M. Mestschersky qui en a fait une intéressante application à un problème traité par Jacobi.

4). Supposons, en général, que X_1, X_2, ... ne contiennent pas t et soient des fonctions homogènes de x_1, x_2, ... d'un degré n différent de -3. Le changement de variables indiqué transforme les équations

$$\frac{d^2x_k}{dt^2} = \phi_k(x_1, x_2, \ldots .)$$

en

$$\frac{d^2y_k}{d\tau^2} + 2a\frac{dy_k}{d\tau} + by_k = \frac{\phi_k(y_1, y_2, \ldots .)}{\mu^2\lambda^{1-n}}.$$

Prenons, d'après (5) et (7),

$$\lambda^2\mu = Ae^{2a\tau},$$

$$\lambda = Be^{(a \pm ic)\tau};$$

nous aurons, pour le produit $\mu^2\lambda^{1-n}$, une constante multipliée par l'exponentielle

$$e^{\tau[(1-n)a \pm i(3+n)c]}.$$

Donc en faisant

$$(1-n)a \pm i(3+n)c = 0, \tag{10}$$

on pourra prendre

$$\mu^2\lambda^{1-n} = 1.$$

Comme $c^2 = b - a^2$, la relation (10) donne

$$(1-n)^2a^2 + (3+n)^2(b-a^2) = 0$$

ou

$$b = \frac{8a^2(n+1)}{(n+3)^2}. \tag{11}$$

Donc si on sait intégrer les équations

$$\frac{d^2x_k}{dt^2} = \phi_k(x_1, x_2, x_3, \ldots.)$$

où ϕ_k est une fonction de $x_1, x_2, \ldots.$, homogène de degré n, on sait intégrer également les équations du mouvement plus général,

$$\frac{d^2y_k}{d\tau^2} + 2a\frac{dy_k}{d\tau} + by_k = \phi_k(y_1, y_2, y_3, \ldots.),$$

b étant lié à a par la relation (11).

Par exemple, l'intégration du problème des deux corps ($n = -2$) conduit à l'intégration du même problème quand s'ajoute une attraction proportionnelle à la distance et une résistance proportionnelle à la vitesse. Alors

$$b = -8a^2.$$

5). Une transformation de même nature pourra être faite sur le mouvement d'un système dépendant de k paramètres $q_1, q_2 \ldots. q_k$, qu'il conviendra de distinguer en paramètres *linéaires*, $q_1, q_2, \ldots. q_h$ représentant des longueurs, et paramètres angulaires $q_{h+1}, q_{h+2}, \ldots. q_k$ répresentant des angles. La demi-force vive T est alors une fonction homogène du second degré de $q_1, q_2, \ldots. q_h$ et de leurs dérivées par rapport au temps t. On obtient des résultats qui sont la généralisation des précédents, en faisant, dans les équations de Lagrange, la transformation

$$q_1 = \lambda p_1,\ q_2 = \lambda p_2, \ldots., q_h = \lambda p_h,$$
$$d\tau = \mu dt.$$

On a ainsi un nouveau mode de transformation venant s'ajouter à ceux qu'on a employés dejà. (Voyez Journal de Crelle, t. 110, 1892, p. 37; voyez également un Mémoire de M. Painlevé, Journal de M. Jordan, 1894.)

Extrait d'une lettre adressée à M. Craig

PAR M. HERMITE.

.... permetter moi de vous offrir pour l'*Americain Journal de Mathématiques* le résultat d'une recherche qui a fait le sujet d'une de mes leçons, sur la valeur asymptotique de $\log \Gamma(a)$ lorsque a est supposé un grand nombre. En étudiant les diverses méthodes par lesquelles a été traitée cette question importante, j'avais déjà remarqué que l'intégrale de Raabe,

$$\int_0^1 \log \Gamma(a+x)\,dx = a \log a - a + \log \sqrt{2\pi},$$

peut y être introduite avec avantage, mais la considération de cette quantité s'offre plus naturellement que je ne l'avais encore vu, sous le point de vue nouveau que je vais vous indiquer.

Je parts de cette identité élémentaire où U et V sont deux fonctions quelconques de x,

$$\int UV''\,dx = UV' - VU' + \int VU''\,dx,$$

puis je fais,

$$U = x - x^2,$$
$$V = F(a+x),$$

a désignant une constante, et je prends les intégrales entre les limites $x = 0$ et $x = 1$. On obtient ainsi la relation,

$$\int_0^1 (x - x^2)\,F''(a+x)\,dx = F(a+1) + F(a) - 2\int_0^1 F(a+x)\,dx.$$

Cela étant soit $F(x) = \log \Gamma(x)$, posons encore,

$$J = \int_0^1 \log \Gamma(a+x)\,dx,$$

$$J(a) = \tfrac{1}{2}\int_0^1 (x - x^2)\,D_x^2 \log \Gamma(a+x)\,dx;$$

au moyen de l'égalité,

$$\log \Gamma(a+1) + \log \Gamma(a) = 2 \log \Gamma(a) + \log a,$$

on trouve l'expression suivante,

$$\log \Gamma(a) = J - \tfrac{1}{2}\log a + J(a).$$

Elle met immédiatement en évidence que la quantité,

$$J - \tfrac{1}{2}\log a = (a - \tfrac{1}{2})\log a - a + \log\sqrt{2\pi},$$

est la valeur approchée du premier membre, pour a très grand. Il est facile en effet d'obtenir une limite supérieure de $J(a)$, si l'on remarque que la facteur $D_x^2 \log \Gamma(a+x)$ qui figure sous le signe d'intégration est toujours positif. C'est ce que montre la série,

$$D_x^2 \log \Gamma(a+x) = \frac{1}{(a+x)^2} + \frac{1}{(a+x+1)^2} + \dots$$

ou encore la formule,

$$D_x^2 \log \Gamma(a+x) = \int_{-\infty}^{0} \frac{ye^{(a+x)y}dy}{e^y - 1},$$

conséquence de l'expression de Cauchy,

$$\log \Gamma(a) = \int_{-\infty}^{0} \left[\frac{e^{ay} - e^y}{e^y - 1} - (a-1)e^y\right]\frac{dy}{y},$$

la quantité $\dfrac{y}{e^y - 1}$ étant positive pour toutes les valeurs de y. Cette limite est donnée par la relation suivante où ξ est compris entre zéro et l'unité,

$$J(a) = \tfrac{1}{2}(\xi - \xi^2)\int_0^1 D_x^2 \log \Gamma(a+x)\,dx = \frac{\xi - \xi^2}{2a}.$$

Le maximum de $\xi - \xi^2$ est $\frac{1}{4}$, on peut donc écrire en désignant par θ un nombre positif, inférieur à l'unité,

$$J(a) = \frac{\theta}{8a}.$$

Nous pouvons même aller plus loin et parvenir à la limite précise

$$J(a) = \frac{\theta}{12a},$$

comme je vais le faire voir.

Je tire pour cela de la relation générale,

$$\int_0^1 F(x)\,dx = \int_0^{\frac{1}{2}} [F(x) + F(1-x)]\,dx,$$

cette nouvelle expression,

$$J(a) = \tfrac{1}{2}\int_0^{\frac{1}{2}} (x - x^2)\, D_x^2 [\log \Gamma(a+x) + \log \Gamma(a+1-x)]\,dx,$$

et je remarque que les quantités $x - x^2$ et $D_x^2\,[\log\Gamma(a+x) + \log\Gamma(a+1-x)]$ varient en sens contraire entre les limites de l'intégrale. La première est croissante tandis que la seconde qui a pour valeur,

$$\int_{-\infty}^{0} \frac{ye^{ay}\,[e^{xy} + e^{(1-x)y}]}{e^y - 1}\,dy$$

est au contraire décroissante, comme on le reconnait au moyen de la dérivée,

$$\int_{-\infty}^{0} \frac{y^2e^{ay}\,[e^{xy} - e^{(1-x)y}]\,dy}{e^y - 1}.$$

En effet, le dénominateur $e^y - 1$ est négatif, et si nous supposons $x < 1 - x$, c'est-à-dire $x < \frac{1}{2}$, on a puisque la variable y est négative, $e^{xy} > e^{(1-x)y}$. Nous pouvons en conséquence appliquer le théorème de M. Tchebichef qui consiste en ce que les fonctions $\phi(x)$ et $\psi(x)$ étant l'une croissante, l'autre décroissante, lorsque la variable croit de a à b, on a

$$(b-a)\int_a^b \phi(x)\,\psi(x)\,dx < \int_a^b \phi(x)\,dx\,.\int_a^b \psi(x)\,dx.$$

Il vient ainsi:

$$J(a) < \int_0^{\frac{1}{2}} (x - x^2)\,dx\,.\int_0^{\frac{1}{2}} D_x^2 \log[\Gamma(a+x)\,\Gamma(a+1-x)]\,dx$$

et en employant la relation dont il a été déjà fais usage,

$$\int_0^1 F(x)\,dx = \int_0^{\frac{1}{2}} [F(x) + F(1-x)]\,dx,$$

nous trouvons sous une forme plus simple,

$$J(a). < \int_0^{\frac{1}{2}} (x - x^2)\,dx\,.\int_0^1 D_x^2 \log\Gamma(a+x)\,dx.$$

On en conclut la limitation à la quelle il s'agissait de parvenir,

$$J(a) = \frac{\theta}{12a}.$$

La valeur de la quantité $J(a)$ que nous avons obtenue au moyen de l'intégrale d'une fonction uniforme, n'ayant que des discontinuités polaires, conduit à des conséquences analytiques importantes. Ayant en effet,

$$D_x^2 \log\Gamma(a+x) = \frac{1}{(a+x)^2} + \frac{1}{(a+x+1)^2} + \ldots\,,$$

on voit que l'expression,

$$\int_0^1 (x - x^2)\,D_x^2 \log\Gamma(a+x)\,dx,$$

sera finie et déterminée avec un sens unique, pour toutes les valeurs réelles ou imaginaires de a, à moins qu'on ne suppose,

$$a + x = 0,\ -1,\ -2,\ \ldots.$$

La variable x croissant dans l'intégrale de zéro à l'unité, les valeurs négatives sont à exclure, nous excepterons par suite la partie illimitée à gauche de l'origine, de l'axe des abscisses, en représentant a par l'affixe d'un point rapporté à des axes rectangulaires dans un plan. J'ajoute que le long de cette ligne la différence,

$$J(a + i\lambda) - J(a - i\lambda)$$

ne tend pas vers zéro, pour une valeur infiniment petite de λ que je supposerai réelle et positive; la partie négative de l'axe des abscisses sera ainsi une coupure pour la fonction $J(a)$.

Soit en effet $a = -n - \xi$, n étant un entier quelconque et ξ une quantité positive moindre que l'unité. J'envisage l'expression qu'on tire de la formule

$$J(a) = \tfrac{1}{2}\int_0^1 (x - x^2)\, D_a^2 \log \Gamma(a + x)\, dx,$$

au moyen de la série qui représente $D_a^2 \log \Gamma(a + x)$, et après avoir remplacé a par $a + i\lambda$, j'observe qu'entre les limites de l'intégrale, tous les termes seront finis pour $\lambda = 0$, sauf un seul qui correspond à la fraction $\dfrac{1}{(a+n+x)^2}$, c'est-à-dire $\dfrac{1}{(x-\xi)^2}$. Les termes finis disparaissent dans la différence de $J(a+i\lambda) - J(a-i\lambda)$ qui sera par conséquent donnée par l'intégrale,

$$\tfrac{1}{2}\int_0^1 (x - x^2)\left[\frac{1}{(x - \xi + i\lambda)^2} - \frac{1}{(x - \xi - i\lambda)^2}\right] dx.$$

On la ramène d'abord si on intègre par parties à la suivante,

$$\tfrac{1}{2}\int_0^1 (1 - 2x)\left[\frac{1}{x - \xi + i\lambda} - \frac{1}{x - \xi - i\lambda}\right] dx,$$

et en simplifiant,

$$i\int_0^1 \frac{\lambda\,(2x - 1)\, dx}{\lambda^2 + (x - \xi)^2}$$

Cela étant faisons $x - \xi = \lambda y$, et l'on trouvera,

$$\int_0^1 \frac{\lambda\,(2x - 1)\, dx}{\lambda^2 + (x - \xi)^2} = \int_{-\frac{\xi}{\lambda}}^{\frac{1-\xi}{\lambda}} \frac{(2\xi - 1 + 2\lambda y)\, dy}{1 + y^2}$$

$$= (2\xi - 1)\left(\operatorname{arctg}\frac{1 - \xi}{\lambda} + \operatorname{arctg}\frac{\xi}{\lambda}\right) + \lambda \log \frac{\lambda^2 + (1 - \xi)^2}{\lambda^2 + \xi^2}.$$

La limite pour λ infiniment petit et positif, est $(2\xi - 1)\pi$; on en conclut la relation

$$J(a + i\lambda) - J(a - i\lambda) = (2\xi - 1)\, i\pi,$$

que je me suis proposé d'obtenir, elle s'écrit habituellement sous la forme,

$$J(\overset{+}{a}) - J(\bar{a}) = (2\xi - 1)\, i\pi,$$

et l'on remarquera le caractère arithmétique de la quantité ξ qui est l'excês de la quantité positive $-a$ sur le nombre entier le plus voisin par défaut.

C'est M. Stieltjes qui a le premier obtenu l'extension de la fonction $J(a)$ à tout le plan, affecté d'une coupure, dans son beau mémoire, *sur le développement de log* $\Gamma(x)$ (Journal de Mathématiques de M. Jordan, t. V, p. 428). Le point de vue auquel je me suis placé est entièrement différent de celui de l'illustre géomêtre, et conduit en suivant une marche inverte à conclure de la nouvelle expression de $J(a)$ qui sert de base et de point de départ, les formules précédemment obtenues par Binet et Gudermann. De l'intégrale,

$$\tfrac{1}{2}\int_0^1 \frac{(x - x^2)\,dx}{(a + x + n)^2} = (a + n + \tfrac{1}{2}) \log\left(1 + \frac{1}{a + n}\right) - 1$$

je tire d'abord la série de Gudermann,

$$J(a) = \Sigma\left[(a + n + \tfrac{1}{2}) \log\left(1 + \frac{1}{a + n}\right) - 1\right] \qquad (n = 0,\ 1,\ 2, \ldots .)$$

L'expression dont j'ai fait usage plus haut,

$$D_x^2 \log \Gamma(a + x) = \int_{-\infty}^{0} \frac{y e^{(a + x) y}\, dy}{e^y - 1}$$

donne ensuite d'une manière facile, en remarquent que l'on a,

$$\int (x - x^2)\, e^{xy} dx = e^{xy} \left(\frac{x - x^2}{y} - \frac{1 - 2x}{y^2} - \frac{2}{y^3}\right)$$

et par conséquent,

$$\int_0^1 (x - x^2)\, e^{xy} dy = \frac{e^y (y - 2) + y + 2}{y^3},$$

la formule de Binet employée par Cauchy dans son mémoire célèbre sur la série de Stirling, à savoir:

$$J(a) = \int_{-\infty}^{0} \frac{e^{ay}\, [e^y (y - 2) + y + 2]\, dy}{2y^2 (e^y - 1)}.$$

En partant de cette expression et par diverses méthodes on parvient enfin à un autre résultat du encore à Binet,

$$J(a) = \frac{1}{\pi} \int_0^{-\infty} \frac{a \log(1 - e^{2\pi x})\, dx}{x^2 + a^2}$$

qui permet comme on sait d'arriver par une voie plus facile et plus simple que celle de Cauchy au reste de cette série de Stirling et à l'importante conclusion obtenue par le grand géomêtre.

On the First and Second Logarithmic Derivatives of Hyperelliptic σ-Functions.

By Oskar Bolza.

Introduction.

Let

$$R(x) = A_0x^4 + 4A_1x^3 + 6A_2x^2 + 4A_3x + A_4$$

be a quartic whose discriminant is different from zero; further,

$$y^2 = R(x), \quad \eta^2 = R(\xi)$$

and

$$w^{x\xi} = \int_{(\xi,\,\eta)}^{(x,\,y)} \frac{dx}{y};$$

then, according to well-known theorems of the theory of elliptic functions,[*]

$$\frac{\sigma'}{\sigma}(w^{x\xi}) = \frac{1}{2}\,\frac{y+\eta}{x-\xi} - \int_{(\xi,\,\eta)}^{(x,\,y)} (A_0x^2 + 2A_1x + A_2)\,\frac{dx}{2y} \tag{a}$$

and[†]

$$\wp(w^{x\xi}) = \frac{y\eta + F(x,\xi)}{2(x-\xi)^2}; \tag{b}$$

$F(x, \xi)$ being the second polar[‡] of ξ with regard to $R(x)$. In particular, a being a root of $R(x)$,[||]

$$\wp(w^{xa}) = \frac{1}{24}R''(a) + \frac{1}{4}\,\frac{R'(a)}{x-a}, \tag{c}$$

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[*] I cannot refer to a place where (a) is given precisely in this form; it follows, however, readily from a formula proved by Weierstrass in one of his courses on Elliptic Functions, viz. If $u = w^{xa}$, $u_0 = w^{x_0a}$, a being a branchpoint, then $\wp(u + u_0) + \wp(u - u_0) = A_0x^2 + 2A_1x + A_2$, or else it may be derived from (b).

[†] Weierstrass, Lectures on Elliptic Functions. See Enneper-Müller, Elliptische Functionen, p. 29. The covariant character of the formula was first recognized by Klein, Hyperelliptische σ-Functionen, Math. Ann., Bd. 27, p. 464; see also Halphen, Traité des fonctions elliptiques, II, p. 359, and Harkness and Morley, Theory of Functions, p. 340.

[‡] That is, if we introduce homogeneous variables, $x = x_1/x_2$, $\xi = \xi_1/\xi_2$ and put $x_2^4 R(x) = a_x^4$, then $F(x, \xi)\,\xi_2^2x_2^2 = a_x^2\,a_\xi^2$.

[||] Weierstrass, Lectures on Elliptic Functions; see Enneper-Müller, l. c., p. 30.

and differentiating

$$\wp'(w^{xa}) = -\frac{1}{4}\,\frac{R'(a)}{(x-a)^2}\,y. \tag{d}$$

If a' be another root of $R(x)$ and

$$\phi(x) = (x-a)(x-a'), \quad R(x) = \phi(x)\,\psi(x),$$

then[*]

$$\wp(w^{aa'}) = -\frac{1}{6}\,(\phi\psi)^2, \tag{e}$$

$(\phi\psi)^2$ denoting the second transvectant of

$$\phi_x^2 = x_2^2\,\phi\left(\frac{x_1}{x_2}\right) \text{ and } \psi_x^2 = x_2^2\,\psi\left(\frac{x_1}{x_2}\right).$$

In the following paper I propose to give an *extension of these and some allied theorems to hyperelliptic functions.*

PART I.

THE FIRST LOGARITHMIC DERIVATIVES.

§1.—*Notations.*

Let

$$y^2 = R(x) \equiv \sum_{i=0}^{2\rho+2} \binom{2\rho+2}{i} A_i x^{2\rho+2-i} \equiv A_0 \prod_{k=0}^{2\rho+} (x-a_k) \tag{1}$$

be the *hyperelliptic curve* under consideration, of deficiency ρ; further,

$$w_\alpha^{x\xi} = \int_{(\xi,\,\eta)}^{(x,\,y)} \frac{g_\alpha(x)\,dx}{y}, \qquad (\alpha = 1, 2, \dots \rho) \tag{2}$$

a set of ρ linearly independent *integrals of the first kind*, the $g_\alpha(x)$'s being integral functions of degree not higher than the $(\rho-1)^{\text{st}}$; and

$$Q_{\xi_1\xi_0}^{x_1x_0} = \int_{(x_0,\,y_0)}^{(x_1,\,y_1)} \int_{(\xi_0,\,\eta_0)}^{(\xi_1,\,\eta_1)} \frac{y\eta + F(x,\xi)}{2(x-\xi)^2\,y\eta}\,dx\,d\xi, \tag{3}$$

Klein's commutative *integral of the third kind*,[†] $F(x,\xi)$ being the $(\rho+1)^{\text{st}}$ polar of ξ with regard to $R(x)$, in the same sense as above.

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[*] Klein, Lectures on Hyperelliptic Functions, 1887. In Math. Ann., Bd. 27, p. 459, (64), the factor $\frac{1}{3}$ should be changed into $-\frac{1}{6}$.

[†] Klein, Hyperelliptische σ-Functionen, Math. Ann., Bd. 32, p. 365. I use, however, non-homogeneous notation, which is better adapted for our present purpose.

The "*normal integrals of the second kind*":[*]

$$w^{x\xi}_{\rho+\alpha} = \int_{(\xi,\,\eta)}^{(x,\,y)} \frac{g_{\rho+\alpha}(x)\,dx}{y}, \qquad (\alpha = 1,\, 2,\, \ldots .\, \rho) \tag{4}$$

associated with the integrals (2) and (3), are determined by the identical relation[†]

$$\frac{d}{d\xi}\,\frac{1}{2}\,\frac{y+\eta}{(x-\xi)\,y} - \sum_{\alpha} \frac{g_\alpha(x)}{y}\,\frac{g_{\rho+\alpha}(\xi)}{\eta} = \frac{y\eta + F(x,\,\xi)}{2\,(x-\xi)^2\,y\eta} \tag{5}$$

which furnishes for the $g_{\rho+\alpha}(x)$'s integral functions each of degree not higher than the 2ρth.[‡]

With a system of canonical periods

$$\begin{matrix} 2\omega_{\alpha\beta}, & 2\omega'_{\alpha\beta}, \\ 2\eta_{\alpha\beta}, & 2\eta'_{\alpha\beta}, \end{matrix} \tag{6}$$

of the integrals (2) and (4) as "moduli," we construct the $2^{2\rho}$ *Θ-functions*[§] of ρ independent variables $u_1,\, u_2,\, \ldots .\, u_\rho$. They differ only by constant factors—which are of no consequence for our present purpose—from Klein's *σ-functions*:[||]

$$\sigma_{\phi\psi}(u_1,\, u_2,\, \ldots .\, u_\rho) = \frac{\Theta_{\left[\substack{g\\h}\right]}(u_1,\, u_2,\, \ldots .\, u_\rho)}{C_{\phi\psi}}. \tag{7}$$

The correspondence between[**] the "transcendental characteristics" $\begin{bmatrix} g \\ h \end{bmatrix}$ and the "algebraic characteristics" $\phi\psi$ depends on the canonical dissection of the Riemann-surface. In order to fix the ideas we suppose the dissection so chosen that the algebraic characteristic $\phi_0\psi_0$ of the "fundamental" Θ-function $\Theta_{\left[\begin{smallmatrix}0 \dots\dots 0\\ 0 \dots\dots 0\end{smallmatrix}\right]}$ is

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[*] Weierstrass, Lectures on Hyperelliptic Functions. See Wiltheiss, Journal für Math., Bd. 99, p. 238; Math. Ann., Bd. 31, p. 136, and Bd. 33, p. 269. I follow here the notation used by Weierstrass in his course of winter 1881–82, which differs slightly from Wiltheiss' notation; accordingly I write $g_\alpha(x)$ and $g_{\rho+\alpha}(x)$ instead of Wiltheiss' $\frac{H_\alpha(x)}{2}$ and $\frac{G_\alpha(x)}{2}$.

[†] As in Weierstrass' lectures, *the letters α, β, γ will always be used to denote indices running from 1 to ρ.*

[‡] The explicit expressions of these integral functions are of no consequence for our investigation; they may, however, be found in my paper "On Weierstrass' systems of hyperelliptic integrals of the first and second kind," read before the Mathematical Congress in Chicago, 1893.

[§] Weierstrass, Lectures; see Schottky, Abel'sche Functionen, §1, and Wiltheiss, ll. cc.

[||] Klein, Math. Ann., Bd. 32, pp. 367, 376; Burkhardt, Math. Ann., Bd. 32, pp. 418, 427; and Schröder, Ueber den Zusammenhang der hyperelliptischen σ- und ϑ-Functionen, Diss. Göttingen, 1890.

[**] See Prym, Functionen in einer zweiblättrigen Fläche, Zürich, 1866, Art. 6; Klein, Lectures on Hyperelliptic Functions, 1887–88, and Thompson, Hyperelliptische Schnittsysteme und Zusammenordnung der algebraischen und transcendentalen Thetacharacteristiken, *American Jour. of Math.*, Vol. XV, p. 91.

$$\begin{aligned}\phi_0(x) &= \text{const.}\,(x-a_1)(x-a_3)\,\dots\dots\,(x-a_{2\rho+1}),\\ \psi_0(x) &= \text{const.}\,(x-a_0)(x-a_2)\,\dots\dots\,(x-a_{2\rho}).\end{aligned} \tag{8}$$

After these preliminaries we *define*

$$\zeta_{\phi\psi}(u_1,\, u_2,\, \dots\dots\, u_\rho;\, t) = \sum_\alpha \frac{\partial \log \sigma_{\phi\psi}(u_1,\, u_2,\, \dots\dots u_\rho)}{\partial u_\alpha}\, g_\alpha(t), \tag{9}$$

t being an auxiliary independent variable. The letter ζ is chosen to indicate the analogy with the elliptic function $\zeta u = \frac{\sigma'}{\sigma}(u)$. In fact, for $\rho = 1$ and $g_1(x) = 1$, we obtain the four functions

$$\frac{\sigma'}{\sigma}(u),\quad \frac{\sigma_1'}{\sigma_1}(u),\quad \frac{\sigma_2'}{\sigma_2}(u),\quad \frac{\sigma_3'}{\sigma_3}(u).$$

The introduction of the auxiliary variable t—which may at first sight seem rather artificial—allows to unite in a very convenient way theorems concerning the *system of ρ functions*,

$$\frac{\partial \log \sigma}{\partial u_1},\quad \frac{\partial \log \sigma}{\partial u_2},\, \dots\dots\, \frac{\partial \log \sigma}{\partial u_\rho}.$$

A last remark concerning notations refers to homogeneous variables. For the greater part of our work it is simpler to use non-homogeneous variables, but sometimes homogeneous variables are preferable. In order to avoid a complication of notations, the same functional symbol $f(x)$ will be used to designate an integral *function* of x, say of degree n, when non-homogeneous variables are used, and the binary *quantic* $x_2^n f\left(\frac{x_1}{x_2}\right)$ when homogeneous variables are used. Thus $\zeta(u_1, \dots\dots u_\rho;\, t)$ will occasionally stand for the homogeneous function of t_1, t_2:

$$t_2^{\rho-1}\zeta\left(u_1,\, \dots\dots\, u_\rho;\, \frac{t_1}{t_2}\right).$$

§2.—*Expression of ζ in terms of the Normal Integrals of the Second Kind.*

We now *replace in $\zeta(u_1,\, u_2,\, \dots\dots\, u_\rho;\, t)$* the independent variables u_α by sums of $\nu \overline{\overline{>}} \rho$ integrals of the first kind,*

$$w_\alpha = w_\alpha^{x_1\xi_1} + w_\alpha^{x_2\xi_2} + \dots\dots\, w_\alpha^{x_\nu\xi_\nu}. \tag{10}$$

---

*We suppress the characteristic $\phi\psi$ where no distinction between different characteristics is necessary.

Our first problem is to express $\zeta(w_1, \ldots . w_\rho; t)$ in terms of the normal integrals of the second kind, $w^{x_i \xi_i}_{\rho+\alpha}$.

We begin with the following

Lemma: If $\pi(t)$ denote the product

$$\pi(t) = \prod_i (t - x_i),$$

then

$$\zeta(w_1, \ldots . w_\rho; t) = \sum_i \frac{\pi(t)}{\pi'(x_i)(t - x_i)}\, y_i \frac{\partial \log \sigma}{\partial x_i}, \tag{11}$$

the index i running from 1 *to* ν.

Proof: Logarithmic differentiation of $\sigma(w_1, \ldots . w_\rho)$ with respect to x_i gives

$$\frac{\partial \log \sigma}{\partial x_i} = \sum_\alpha \frac{\partial \log \sigma}{\partial w_\alpha} \frac{g_\alpha(x_i)}{y_i},$$

or, according to (9):

$$y_i \frac{\partial \log \sigma}{\partial x_i} = \zeta(w_1, \ldots . w_\rho; x_i).$$

But since $\zeta(w_1, \ldots . w_\rho; t)$ is an integral function of t of degree $\rho - 1$, and $\nu \geqq \rho$, we have, by decomposition into partial fractions,

$$\frac{\zeta(w_1, \ldots . w_\rho; t)}{\pi(t)} = \sum_i \frac{\zeta(w_1, \ldots . w_\rho; x_i)}{\pi'(x_i)(t - x_i)} = \sum_i \frac{y_i \dfrac{\partial \log \sigma}{\partial x_i}}{\pi'(x_i)(t - x_i)}. \qquad \text{Q. E. D.}$$

Corollary: Evidently (11) remains true if in the sum $\sum\limits_i$ and in the product $\prod\limits_i$ the index i takes—not all but—*at least* ρ *of the* ν *values* 1, 2, ... ν. More generally, if $h(t)$ denote an arbitrary polynomial whose degree is $\leqq \nu - \rho$, then

$$\zeta(w_1, \ldots . w_\rho; t) = \sum_{i=1}^{\nu} \frac{\pi(t)\, h(x_i)}{\pi'(x_i)(t - x_i)\, h(t)}\, y_i \frac{\partial \log \sigma}{\partial x_i}. \tag{12}$$

Our problem is now reduced to the computation of $\dfrac{\partial \log \sigma}{\partial x_i}$; to this effect we start from Klein's explicit expression of $\sigma_{\phi\psi}(w_1, \ldots . w_\rho)$ in terms of $x_1, \ldots . x_\nu;\ \xi_1, \ldots . \xi_\nu$:[*]

Let

$$\Omega(x, \xi) = \frac{(x - \xi)}{\sqrt{y\eta}}\, e^{\frac{1}{2} Q^{x\xi}_{x\xi}}, \tag{13}$$

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[*] **Klein, Hyperelliptische σ-Functionen, Math. Ann., Bd. 32, p. 367.**

$\bar{x}$ and $\bar{\xi}$ denoting the points $(x, -y)$ and $(\xi, -\eta)$ conjugate with the points $x = (x, y)$ and $\xi = (\xi, \eta)$.

Further, if $\rho + 1 - 2\mu$ and $\rho + 1 + 2\mu$ are the respective degrees of ϕ and ψ, let $D_{\phi\psi}$ denote the determinant of order 2ν:

$$D_{\phi\psi} = \tag{14}$$

$$\left| \begin{matrix} \sqrt{\phi(x_i)}, & x_i\sqrt{\phi(x_i)} \ldots. & x_i^{\nu-1+\mu}\sqrt{\phi(x_i)}; & \sqrt{\psi(x_i)}, \; x_i\sqrt{\psi(x_i)} \ldots. \\ & & & \ldots. x_i^{\nu-1-\mu}\sqrt{\psi(x_i)} \\ -\sqrt{\phi(\xi_i)}, & -\xi_i\sqrt{\phi(\xi_i)} \ldots. & -\xi_i^{\nu-1+\mu}\sqrt{\phi(\xi_i)}; & \sqrt{\psi(\xi_i)}, \; \xi_i\sqrt{\psi(\xi_i)} \ldots. \\ & & & \ldots. \xi_i^{\nu-1-\mu}\sqrt{\psi(\xi_i)}, \end{matrix} \right|$$

$$i = 1, 2, \ldots. \nu,$$

then

$$\sigma_{\phi\psi}(w_1, \ldots. w_\rho) = c_\nu^{(\mu)} \frac{D_{\phi\psi} \prod_i \prod_k \Omega(x_i, \xi_k)}{\prod_i \prod_k (x_i - \xi_k) \prod_{(i,k)} \Omega(x_i, x_k) \prod_{(i,k)} \Omega(\xi_i, \xi_k)}; \tag{15}$$

$c_\nu^{(\mu)}$ is a numerical constant which has no influence upon our conclusions; in the double products $\prod_i \prod_k$, i and k take independently the values $1, 2, \ldots. \nu$, whereas the products $\prod_{(i,k)}$ extend over the $\frac{\nu(\nu-1)}{2}$ two-combinations of the numbers $1, 2, \ldots. \nu$.

By logarithmic differentiation of (15) we get

$$\frac{\partial \log \sigma}{\partial x_i} = \frac{\partial \log D}{\partial x_i} - \sum_k \frac{1}{x_i - \xi_k} + \sum_k \frac{\partial \log \Omega(x_i, \xi_k)}{\partial x_i} - \sum_k{}' \frac{\partial \log \Omega(x_i, x_k)}{\partial x_i}, \tag{16}$$

where $\sum_k$ extends over all the values $1, 2, \ldots. \nu$, $\sum_k{}'$ over all the values $1, 2, \ldots. \nu$ with the exception of $k = i$.

But from (13) and (3) follows

$$\frac{\partial \log \Omega(x_i, x_k)}{\partial x_i} = \frac{1}{x_i - x_k} - \frac{1}{4} \frac{R'(x_i)}{R(x_i)} + \frac{1}{2} \frac{\partial}{\partial x_i} Q_{x_i x_k}^{\bar{x}_i \bar{x}_k},$$

and

$$\frac{\partial \bar{Q}_{x_i x_k}^{\bar{x}_i x_k}}{\partial x_i} = \int_{\bar{x}_k}^{\bar{x}_i} \frac{y_i \eta + F(x_i, \xi)}{2(x_i - \xi)^2 y_i \eta} d\xi + \int_{x_k}^{x_i} \frac{\bar{y}_i y + F(\bar{x}_i, x)}{2(\bar{x}_i - x)^2 \bar{y}_i y} dx,$$

and since $\bar{x}_i = x_i$, $\bar{y}_i = -y_i$,

$$\frac{\partial Q_{x_i x_k}^{\bar{x}_i \bar{x}_k}}{\partial x_i} = 2 \int_{k}^{x_i} \frac{y_i y - F(x_i, x)}{2(x_i - x)^2 y_i y} dx.$$

But according to (5),

$$\frac{y_i y - F(x_i,\,x)}{2\,(x_i - x)^2\,y_i y} = \frac{d}{dx}\,\frac{1}{2}\,\frac{y - y_i}{(x - x_i)\,y_i} + \sum_{\alpha} \frac{g_\alpha(x_i)}{y_i}\,\frac{g_{\rho+\alpha}(x)}{y}\,;$$

hence integrating,

$$\int_{x_k}^{x_i} \frac{y_i y - F(x_i,\,x)}{2\,(x_i - x)^2\,y_i y}\,dx = \left[\frac{1}{2}\,\frac{y - y_i}{(x - x_i)\,y_i}\right]_{x_k}^{x_i} + \sum_{\alpha} \frac{g_\alpha(x_i)}{y_i}\,w_{\rho+\alpha}^{x_i x_k},$$

and since

$$\left[\frac{1}{2}\,\frac{y - y_i}{(x - x_i)\,y_i}\right]_{x = x_i} = \frac{1}{2}\,\frac{R'(x_i)}{R(x_i)},$$

we get

$$\frac{\partial \log \Omega\,(x_i x_k)}{\partial x_i} = \frac{y_i + y_k}{2\,(x_i - x_k)\,y_i} + \sum_{\alpha} \frac{g_\alpha(x_i)}{y_i}\,w_{\rho+\alpha}^{x_i x}. \tag{17}$$

Returning to (16) and writing

$$w_{\rho+\alpha} = w_{\rho+\alpha}^{x_1 \xi_1} + w_{\rho+\alpha}^{x_2 \xi_2} + \ldots\ldots + w_{\rho+\alpha}^{x_\nu \xi_\nu}, \tag{18}$$

we obtain

$$y_i \frac{\partial \log \sigma_{\phi\psi}}{\partial x_i} = y_i \frac{\partial \log D_{\phi\psi}}{\partial x_i} - \frac{1}{2}\sum_k \frac{y_i - \eta_k}{(x_i - \xi_k)} - \frac{1}{2}{\sum_k}' \frac{y_i + y_k}{x_i - x_k} + \sum_{\alpha} g_\alpha(x_i)\,w_{\rho+\alpha}. \tag{19}$$

Substituting this value in (11) and remembering that

$$\sum_i \frac{\pi(t)\,g_\alpha(x_i)}{\pi'(x_i)(t - x_i)} = g_\alpha(t),$$

we obtain the result:

Theorem: If

$$w_\alpha = w_\alpha^{x_1 \xi_1} + w_\alpha^{x_2 \xi_2} + \ldots\ldots + w_\alpha^{x_\nu \xi_\nu},$$

and

$$w_{\rho+\alpha} = w_{\rho+\alpha}^{x_1 \xi_1} + w_{\rho+\alpha}^{x_2 \xi_2} + \ldots\ldots + w_{\rho+\alpha}^{x_\nu \xi_\nu},$$

then

$$\zeta_{\phi\psi}(w_1 \ldots\ldots w_\rho\,;\ t) = \sum_{\alpha} g_\alpha(t)\,w_{\rho+\alpha}$$
$$+ \sum_i \frac{\pi(t)}{\pi'(x_i)(t - x_i)} \left\{ y_i \frac{\partial \log D_{\phi\psi}}{\partial x_i} - \frac{1}{2}\sum_k \frac{y_i - \eta_k}{x_i - \xi_k} - \frac{1}{2}{\sum_k}' \frac{y_i + y_k}{x_i - x_k} \right\}. \tag{A}$$

§3.—*Invariantive Properties of* ζ.

The function $\zeta(w_1, \ldots . w_\rho; t)$ is a *covariant* with respect to a cogredient linear transformation of the variables $x_1, x_2, \ldots . x_\nu; \xi_1, \xi_2, \ldots . \xi_\nu; t$. In fact, apply to the hyperelliptic curve (1) the transformation

$$x = \frac{\alpha x' + \beta}{\gamma x' + \delta} \qquad y = \frac{y'}{(\gamma x' + \delta)^{\rho+1}} \tag{20}$$
$$\alpha\delta - \beta\gamma = r \neq 0,$$

which changes (1) into

$$y'^2 = R_1(x') \equiv (\gamma x' + \delta)^{2\rho+2} R(x). \tag{21}$$

According to a fundamental theorem due to Klein,* σ is a covariant with respect to this transformation; that is to say, let

$$(x'_1, y'_1), (x'_2, y'_2), \ldots . (x'_\nu, y'_1); (\xi'_1, \eta'_1), \ldots . (\xi'_\nu, \eta'_\nu)$$

be the points into which the limits

$$(x_1 y_1), (x_2 y_2), \ldots . (x_\nu y_\nu); (\xi_1 \eta_1), \ldots . (\xi_\nu \eta_\nu)$$

are changed by the transformation (20); further,

$$\phi_1(x') = (\gamma x' + \delta)^{\rho+1-2\mu} \phi(x),$$
$$\psi_1(x') = (\gamma x' + \delta)^{\rho+1+2\mu} \psi(x);$$

then if we denote by $\sigma'_{\phi_1\psi_1}$ the σ-function derived from the curve (20), the limits x'_i, ξ'_i and the characteristic $\phi_1\psi_1$ precisely in the same way as $\sigma_{\phi\psi}$ is derived from the curve (20), the limits x_i, ξ_i and the characteristic $\phi\psi$:†

$$\sigma_{\phi\psi} = r^{\mu^2} \sigma'_{\phi_1\psi_1}. \tag{22}$$

Hence by logarithmic differentiation,

$$y_i \frac{\partial \log \sigma_{\phi\psi}}{\partial x_i} = \frac{r^{-1}}{(\gamma x'_i + \delta)^{\rho-1}} y'_i \frac{\partial \log \sigma'_{\phi_1\psi_1}}{\partial x'_i},$$

and if we put

$$t = \frac{\alpha t' + \beta}{\gamma t' + \delta},$$

and

$$\pi_1(t') = \prod_i (t' - x'_i),$$

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* Math. Ann., Bd. 32, p. 370.

† Observe that the expression (15) of $\sigma_{\phi\psi}(w_1, \ldots w_\rho)$ in terms of $x_1, \ldots x_\nu; \xi_1, \ldots \xi_\nu$ is independent of the choice of the function $g_\alpha(x)$.

we obtain

$$\sum_i \frac{\pi(t)}{\pi'(x_i)(t-x_i)} y_i \frac{\partial \log \sigma_{\phi\psi}}{\partial x_i} = \frac{r^{-1}}{(\gamma t'+\delta)^{\rho-1}} \sum_i \frac{\pi_1(t')(\gamma x'_i+\delta)^{\nu-\rho}}{\pi'_1(x'_i)(t'-x'_i)(\gamma t+\delta)^{\nu-\rho}} \frac{y'_i \partial \log \sigma'_{\phi_1\psi_1}}{\partial x'_i}.$$

But according to (11) and (12) this is

$$\zeta_{\phi\psi}(w_1, \ldots . w_\rho;\ t) = \frac{r^{-1}}{(\gamma t'+\delta)^{\rho-1}} \zeta'_{\phi_1\psi_1}(w'_1, \ldots . w'_\rho;\ t'), \tag{23}$$

If, finally, we introduce homogeneous variables and employ the abbreviated notation agreed upon at the end of §1, equation (23) takes the simple form

$$\zeta_{\phi\psi}(w_1, \ldots . w_\rho;\ t) = r^{-1}\zeta'_{\phi_1\psi_1}(w'_1, \ldots . w'_\rho;\ t), \tag{24}$$

and we may therefore say:

The function $\zeta(w_1, \ldots . w_\rho;\ t)$ *is a covariant of* $R(x)$ *of weight* -1, *with the* $2\nu+1$ *sets of cogredient variables* $x_1x_2 \ldots . x_\nu;\ \xi_1\xi_2 \ldots . \xi_\nu;\ t$.

§4.—*Generalization of Theorem* (*a*).

Formula (A) simplifies considerably under the following two restricting assumptions:

In the first place we suppose $\mu = 0$, and in order to fix the ideas we confine ourselves to the "fundamental" σ-function whose algebraic characteristic $\phi_0\psi_0$ is given by (8); adopting a notation occasionally used by Professor Klein in his lectures, we designate it by σ_0.

In the second place we assume $\nu = \rho + 1$ and choose for the lower limits ξ_k the roots a_{2k-1} of ϕ_0, so that

$$w_\alpha = w_\alpha^{x_1a_1} + w_\alpha^{x_2a_3} + \ldots . + w_\alpha^{x_{\rho+1}a_{2\rho+1}}. \tag{25}$$

These assumptions reduce $D_{\phi\psi}$ to

$$D_0 = \text{const.} \prod_i \sqrt{\phi_0(x_i)} \prod_{(i,\,k)} (x_i - x_k);$$

hence

$$\frac{\partial \log D_0}{\partial x_i} = \frac{1}{2}\,\frac{\phi'_0(x_i)}{\phi_0(x_i)} + \sum_k{}' \frac{1}{x_i - x_k},$$

and

$$y_i \frac{\partial \log \sigma_0}{\partial x_i} = \frac{1}{2}\sum_k{}' \frac{y_i - y_k}{x_i - x_k} + \sum_\alpha g_\alpha(x_i)\, w_{\rho+\alpha}. \tag{26}$$

Substituting this value in (A), we obtain the

Theorem: *If*

$$w_\alpha = w_\alpha^{x_1a_1} + w_\alpha^{x_2a_3} + \ldots . + w_\alpha^{x_{\rho+1}a_{2\rho+1}},$$

and

$$w_{\rho+\alpha} = w_{\rho+\alpha}^{x_1a_1} + w_{\rho+\alpha}^{x_2a_3} + \ldots . + w_{\rho+\alpha}^{x_{\rho+1}a_{2\rho+1}},$$

then

$$\zeta_0(w_1, \ldots\ldots w_\rho;\, t) = \sum_\alpha g_\alpha(t)\, w_{\rho+\alpha} + \sum_i \frac{\pi(t)}{\pi'(x_i)(t-x_i)} \sum_k{}' \frac{1}{2}\frac{y_i - y_k}{x_i - x_k}, \quad (\mathrm{A}_0)$$

where

$$\pi(t) = \prod_i (t - x_i),$$

the index i taking, ad libitum, either all the values $1, 2, \ldots\ldots \rho+1$, *or** ρ *of them, while k takes all the values* $1, 2, \ldots\ldots \rho+1$ *with the exception of* $k = i$.

This is the extension of theorem (a) mentioned in the introduction. In fact, for $\rho = 1$ and $g_1(x) = 1$, the relation (5) yields for $g_2(x)$ the value

$$g_2(x) = -\tfrac{1}{2}(A_0x^2 + 2A_1x + A_2),$$

and (A_0) becomes, if i is allowed to take the value 1 only,

$$\frac{\sigma_0'}{\sigma_0}(w_1) = \frac{1}{2}\frac{y_1 - y_2}{x_1 - x_2} + w_2.$$

Put $x_2 = x_0$, $y_2 = -y_0$, and let

$$\int_{a_2}^{a_1}\frac{dx}{y} = \omega, \quad \int_{a_2}^{a_1}\frac{g_2(x)\,dx}{y} = \eta,$$

then

$$w_1 = \int_{(x_0y_0)}^{(x_1y_1)}\frac{dx}{y} + \omega, \quad w_2 = \int_{(x_0y_0)}^{(x_1y_1)}\frac{g_2(x)\,dx}{y} + \eta.$$

On the other hand

$$\frac{\sigma_0'}{\sigma_0}(u + \omega) = \frac{\sigma'}{\sigma}(u) + \eta.$$

Thus we obtain

$$\frac{\sigma'}{\sigma}\left(\int_{(x_0y_0)}^{(x_1y_1)}\frac{dx}{y}\right) = \frac{1}{2}\frac{y_1 + y_0}{x_1 - x_0} - \frac{1}{2}\int_{(x_0y_0)}^{(x_1y_1)}(A_0x^2 + 2A_1x + A_2)\frac{dx}{y},$$

which is precisely theorem (a).

From (A_0) follows as a special case the expression of a sum of ρ normal integrals of the second kind in terms of the fundamental Θ-function, given by Wiltheiss, Journal für Math., Bd. 99, p. 247.

For if we put, in A_0, $x_{\rho+1} = a_{2\rho+1}$ and write

$$\Phi(t) = (t - x_1)(t - x_2)\ldots\ldots(t - x_\rho), \quad (27)$$

$$u_\alpha = w_\alpha^{x_1a_1} + w_\alpha^{x_2a_3} + \ldots\ldots + w_\alpha^{x_\rho a_{2\rho-1}}, \quad (28)$$

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* Compare the corollary to the lemma of §2.

we have

$$\sum_{\alpha} g_{\alpha}(t)\frac{\partial \log \Theta}{\partial u_{\alpha}} = \sum_{\alpha,\beta} g_{\alpha}(t)\, w_{\rho+\alpha}^{x_{\beta}a_{2\beta-1}}$$

$$+\sum_{\beta}\frac{\Phi(t)}{\Phi'(x_{\beta})(t-x_{\beta})}\left\{\frac{1}{2}\,\frac{y_{\beta}}{x_{\beta}-a_{2\rho+1}}+{\sum_{\gamma}}'\,\frac{1}{2}\,\frac{y_{\beta}-y_{\gamma}}{x_{\beta}-x_{\gamma}}\right\}, \qquad (29)$$

where the index γ takes the values $1, 2, \dots \rho$ with the exception of $\gamma=\beta$.

On the other hand, Wiltheiss' formula, written in our notation, is

$$\sum_{\alpha} g_{\alpha}(t)\frac{\partial \log \Theta}{\partial u_{\alpha}} = \sum_{\alpha,\beta} g_{\alpha}(t)\, w_{\rho+\alpha}^{x_{\beta}a_{2\beta}-1}$$

$$+\sum_{\beta}\frac{1}{2}\,\frac{y_{\beta}}{x_{\beta}-t}-\sum_{\alpha,\beta}\left(\frac{1}{2}\,\frac{1}{x_{\beta}-t}-\frac{1}{2(x_{\beta}-a_{2\rho+1})}\right)g_{\alpha}(t)\frac{\partial x_{\beta}}{\partial u_{\alpha}}.$$

Both formulæ agree exactly if we make use of the relation

$$\sum_{\alpha} g_{\alpha}(t)\frac{\partial x_{\beta}}{\partial u_{\alpha}} = \frac{\Phi(t)\, y_{\beta}}{\Phi'(x_{\beta})(t-x_{\beta})},$$

which follows from the definition of u_{α}, and of the identity

$$\sum_{\beta}\frac{y_{\beta}}{2(x_{\beta}-t)}+\sum_{\beta}\frac{1}{2}\,\frac{\Phi(t)\, y_{\beta}}{\Phi'(x_{\beta})(t-x_{\beta})^{2}}=\sum_{\beta}\frac{\Phi(t)}{2\Phi'(x_{\beta})(t-x_{\beta})}{\sum_{\gamma}}'\frac{y_{\beta}-y_{\gamma}}{x_{\beta}-x_{\gamma}}.$$

To prove the latter, observe that both sides are integral functions of t of degree $\rho-1$ which coincide for the ρ values $t=x_1, x_2, \dots x_{\rho}$.

Lastly, we refer to Thomae's researches on integrals of the second kind, Jour. für Math., Bd. 71, p. 213, and Bd. 93, p. 80, where an allied problem is treated.

PART II.

THE SECOND LOGARITHMIC DERIVATIVES.

§5.—*The Analogue of* $\wp u$ *for* $\rho > 1$.

Before proceeding to the generalization of theorem (b), we must agree what we are to consider as the analogue of $\wp u$ in the hyperelliptic case. Though various ways of generalization are possible, certain results obtained by Brioschi and Wiltheiss, combined with Klein's general theory of hyperelliptic σ-functions, seem to leave no doubt as to the best way.

Brioschi* has given for $\rho = 2$ generalizations of the formula

$$\wp u - e_\lambda = \frac{\sigma_\lambda^2 u}{\sigma^2 u},$$

which were extended to the general hyperelliptic case by Wiltheiss.† In their formulæ the mixed-polar-like expression

$$-\sum_{\alpha,\beta} \frac{\partial^2 \log \Theta}{\partial u_\alpha \, \partial u_\beta} g_\alpha(a_\lambda) \, g_\beta(a_\mu),$$

a_λ, a_μ being two branchpoints, takes the place of $\wp u$.

On the other hand, Wiltheiss‡ has given a generalization for $\rho = 2$ of the differential equations

$$\wp'^2 u = 4\wp^3 u - g_2 \wp u - g_3,$$
$$\wp'' u = 6\wp^2 u - \tfrac{1}{2} g_2.$$

In his results $\wp u$ is replaced by

$$-\sum_{\alpha,\beta} \frac{\partial^2 \log \Theta}{\partial u_\alpha \, \partial u_\beta} g_\alpha(t) \, g_\beta(t),$$

t being an auxiliary variable, and the same expression occurs incidentally in his researches on the partial differential equations of the hyperelliptic Θ-functions.||

These results point to the expression

$$-\sum_{\alpha,\beta} \frac{\partial^2 \log \Theta}{\partial u_\alpha \, \partial u_\beta} g_\alpha(s) \, g_\beta(t),$$

s and t being two auxiliary variables,§ as the true generalization of $\wp u$.

Accordingly we *define*

$$\left.\begin{aligned} \wp_{\phi\psi}(u_1 \ldots . u_\rho;\, s,\, t) &= -\sum_{\alpha,\beta} \frac{\partial^2 \log \sigma_{\phi\psi}(u_1 \ldots . u_\rho)}{\partial u_\alpha \, \partial u_\beta} g_\alpha(s) \, g_\beta(t) \\ &= -\sum_{\alpha} \frac{\partial \zeta_{\phi\psi}(u_1 \ldots . u_\rho;\, t)}{\partial u_\alpha} g_\alpha(s). \end{aligned}\right\} \qquad (30)$$

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* Rendic. dell' Accademia dei Lincei, 1886_1, p. 199.

† Math. Ann., Bd. 31, p. 417.

‡ Jahresbericht der deutschen Mathematiker-Vereinigung, Bd. I, p. 72.

|| Math. Ann., Bd. 31, p. 153, and Bd. 33, p. 286.

§ The introduction of *two* auxiliary variables was suggested to me by Professor Klein.

For $\rho = 1$ and $g_1(x) = 1$ we obtain the four functions

$$\wp u = -\frac{d^2 \log \sigma}{du^2}, \quad \wp(u + \omega_\lambda) = -\frac{d^2 \log \sigma_\lambda}{du^2}, \qquad \lambda = 1, 2, 3.$$

Every $\wp$-function is reducible to the "*fundamental*" *$\wp$-function* whose characteristic is given by (8), and which we shall denote by $\wp_0$, by means of a relation of the form

$$\wp_{\phi\psi}(u_1 + \omega_1, \ldots . u_\rho + \omega_\rho; s, t) = \wp_0(u_1, \ldots . u_\rho; s, t), \tag{31}$$

$\omega_1, \omega_2, \ldots . \omega_\rho$ being a certain system of simultaneous half-periods of the integrals (2).

In the same sense as $\zeta(w_1, \ldots . w_\rho; t)$ in §3, also $\wp(w_1, \ldots . w_\rho; s, t)$ is a *covariant* of $R(x)$, but *of weight* -2; this is easily seen by a repetition of the conclusions of §3.

Concerning the use of the auxiliary variables s and t, an analogous remark applies as in §1; they serve to unite in a simple way theorems concerning the *system of* $\frac{\rho(\rho+1)}{2}$ *functions,*

$$\frac{\partial^2 \log \sigma}{\partial u_\alpha \partial u_\beta}.$$

§6.—*An Interpolation Formula.*

For our further developments we need the following *Lemma: Let $G(s, t)$ be an integral symmetric function of s and t, of degree $\rho - 1$ with respect to each of the two variables; further, $x_1, x_2, \ldots . x_\nu$ $\nu \overline{\overline{>}} \rho + 1$ distinct values, and*

$$\pi(t) = (t - x_1)(t - x_2) \ldots . (t - x_\nu),$$

then
$$G(s, t) = -\sum_{(i,k)} \frac{(x_i - x_k)^2 G(x_i, x_k)\, \pi(s)\, \pi(t)}{\pi'(x_i)\, \pi'(x_k)(t - x_i)(t - x_k)(s - x_i)(s - x_k)}, \tag{32}$$

the summation extending over the $\frac{\nu(\nu-1)}{2}$ two-combinations of the numbers $1, 2, \ldots . \nu$.

Proof: The function $G(x_i, t)$ is an integral function of t of degree $\rho - 1$; the function

$$\pi_i(t) = \frac{\pi(t)}{t - x_i}$$

is at least of degree ρ; hence by decomposition into partial fractions

$$\frac{G(x_i, t)}{\pi_i(t)} = \sum_k{}' \frac{G(x_i, x_k)}{\pi_i'(x_k)(t - x_k)},$$

the index k taking the values $1, 2, \ldots . \nu$ with the exception of $k = i$. But

$$\pi_i'(x_k) = \frac{\pi'(x_k)}{x_k - x_i};$$

hence

$$G(x_i, t) = \sum_k{}' \frac{(x_k - x_i)\, G(x_i, x_k)\, \pi(t)}{\pi'(x_k)(t - x_i)(t - x_k)}.$$

Next consider the function $G(s, t)$ as a function of s alone; again, by decomposition into partial fractions,

$$\frac{G(s, t)}{\pi(s)} = \sum_i \frac{G(x_i, t)}{\pi'(x_i)(s - x_i)},$$

the index i running from 1 to ν.

Substitute the above value of $G(x_i, t)$, observe that generally

$$\sum_i \sum_k{}' c_{ik} = \sum_{(i,k)} (c_{ik} + c_{ki}),$$

and make use of our hypothesis

$$(x_i, x_k) = G(x_k, x_i);$$

thus we obtain

$$\frac{G(s, t)}{\pi(s)} = \sum_{(i,k)} \frac{(x_k - x_i)\, G(x_i, x_k)\, \pi(t)}{\pi'(x_i)\, \pi'(x_k)(t - x_i)(t - x_k)} \left[\frac{1}{s - x_i} - \frac{1}{s - x_k}\right],$$

which leads immediately to (32).

§7.—*Generalization of Theorem* (b).

As in §2, we *replace the independent variables* u_a *of* $\wp(u_1, \ldots . u_\rho; s, t)$ *by sums of* ν *integrals of the first kind,*

$$w_a = w_a^{x_1 \xi_1} + w_a^{x_2 \xi_2} + \ldots . + w_a^{x_\nu \xi_\nu}, \tag{10}$$

supposing, however, $\nu \geqq \rho + 1$.

$\wp(w_1, \ldots . w_\rho; s, t)$ is an integral symmetric function of s and t, of degree $\rho - 1$ in each of the two variables. Therefore the Lemma of last section is applicable, by which the computation of $\wp(w_1, \dots . w_\rho;\, s, t)$ is reduced to the computation of the $\frac{\nu(\nu-1)}{2}$ special values

$$\wp(w_1, \dots . w_\rho;\, x_i, x_k) \qquad (\varkappa \neq i).$$

But from (27) follows

$$\wp(w_1, \dots . w_\rho;\, x_i, x_k) = -\sum_{\alpha,\beta} \frac{\partial^2 \log \sigma}{\partial w_\alpha \partial w_\beta} g_\alpha(x_i)\, g_\beta(x_k) = -y_i y_k \frac{\partial^2 \log \sigma}{\partial x_i \partial x_k}. \tag{33}$$

For our further developments we make again the same restricting assumptions as in §4, viz. we confine ourselves to the fundamental $\wp$-function $\wp_0$ and choose

$$w_a = w_a^{x_1 a_1} + w_a^{x_2 a_2} + \dots . + w_a^{x_{\rho+1} a_{2\rho+1}}. \tag{25}$$

Differentiating (26) with respect to x_k, we get

$$\left.\begin{aligned} y_i y_k \frac{\partial^2 \log \sigma_0}{\partial x_i \partial x_k} &= y_k \frac{\partial}{\partial x_k}\, \frac{y_i - y_k}{x_i - x_k} + \sum_a g_a(x_i)\, g_{\rho+a}(x_k), \\ \text{or, according to (5),} \qquad & \\ &= \frac{y_i y_k - F(x_i, x_k)}{2(x_i - x_k)^2}. \end{aligned}\right\} \tag{34}$$

Thus we obtain the

Theorem: If

$$w_a = w_a^{x_1 a_1} + w_a^{x_2 a_2} + \dots . + w_a^{x_{\rho+1} a_{2\rho+1}},$$

then

$$\wp_0(w_1, w_2, \dots . w_\rho;\, x_i, x_k) = \frac{-y_i y_k + F(x_i, x_k)}{2(x_i - x_k)^2} \tag{B}$$

and

$$\wp_0(w_1, w_2, \dots . w_\rho;\, s, t) = \sum_{(i,k)} \frac{\big(y_i y_k - F(x_i, x_k)\big)\, \pi(s)\, \pi(t)}{2\pi'(x_i)\, \pi'(x_k)(s - x_i)(s - x_k)(t - x_i)(t - x_k)}, \tag{B'}$$

the summation extending over the $\frac{\rho(\rho+1)}{2}$ *two-combinations of the numbers* $1, 2, \dots . \rho + 1$. This is an extension of theorem (b). In fact, for $\rho = 1$ and $g_1(x) = 1$, (B) reduces to

$$\wp_0(w_1) = \frac{-y_1 y_2 + F(x_1, x_2)}{2(x_1 - x_2)^2},$$

where

$$w_1 = w_1^{x_1 a_1} + w_1^{x_2 a_2}.$$

But if we put again, as in §4, $x_2 = x_0$, $y_2 = -y_0$ and

$$\omega = \int_{a_2}^{a_1} \frac{dx}{y},$$

we have
$$\wp_0(w_1) = \wp_0(w_1^{x_1 x_0} + \omega) = \wp(w_1^{x_1 x_0}),$$

whence
$$\wp(w_1^{x_1 x_0}) = \frac{y_0 y_1 + F(x_0, x_1)}{2(x_0 - x_1)^2}. \qquad \text{Q. E. D.}$$

Second proof of (B): We add another proof of (B) which does not presuppose (26), but starts directly from the well-known expression of the quotient of two products of Θ- (or σ-) functions in terms of integrals of the third kind. Let

$$u_\alpha = w_\alpha^{x_1 a_1} + w_\alpha^{x_2 a_2} + \dots\dots + w_\alpha^{x_p a_{2p-1}},$$
$$u_\alpha' = w_\alpha^{\xi_1 a_1} + w_\alpha^{\xi_2 a_2} + \dots\dots + w_\alpha^{\xi_p a_{2p-1}},$$

then*
$$\log \frac{\sigma_0((w_\alpha^{x a_{2p+1}} - u_\alpha))\,\sigma_0((w_\alpha^{\xi a_{2p+1}} - u_\alpha'))}{\sigma_0((w_\alpha^{x a_{2p+1}} - u_\alpha'))\,\sigma_0((w_\alpha^{\xi a_{2p+1}} - u_\alpha))} = \sum_\beta Q_{x\xi}^{x_\beta \xi_\beta}. \tag{35}$$

Hence, differentiating with respect to x and x_β,

$$\frac{\partial^2 \log \sigma_0((w_\alpha^{x a_{2p+1}} - u_\alpha))}{\partial x\, \partial x_\beta} = \frac{\partial^2 Q_{x\xi}^{x_\beta \xi_\beta}}{\partial x\, \partial x_\beta} = \frac{y y_\beta + F(x, x_\beta)}{2(x - x_\beta)^2 y y_\beta}.$$

Replacing the point $x = (x, y)$ by $\overline{x}_{p+1} = (x_{p+1}, -y_{p+1})$, we get

$$\wp_0(w_1, \dots\dots w_p;\, x_\beta, x_{p+1}) = \frac{-y_\beta y_{p+1} + F(x_\beta, x_{p+1})}{2(x_\beta - x_{p+1})^2},$$

$w_1, \dots\dots w_p$ having the same meaning as in (B). Hence follows by a permutation of the letters $x_1, x_2, \dots\dots x_{p+1}$, equation (B), which thus appears as an immediate consequence of (35).

Other form of (B′): Formula (B′) may be thrown into another shape which is better adapted for applications. Add and subtract on the right-hand side

$$\sum_i \frac{R(x_i)\,\pi(s)\,\pi(t)}{4\pi'^2(x_i)(s - x_i)^2(t - x_i)^2}$$

and observe that
$$R(x_i) = y_i^2 = F(x_i, x_i)$$

and
$$\sum_i \sum_k c_{ik} = \sum_i c_{ii} + 2\sum_{(i,k)} c_{ik},$$

provided $c_{ik} = c_{ki}$. The right-hand side of (B′) becomes

$$\pi(s)\,\pi(t)\Big[\sum_i \frac{y_i}{2(s - x_i)(t - x_i)\,\pi'(x_i)}\Big]^2$$
$$- \sum_i \sum_k \frac{F(x_i, x_k)\,\pi(s)\,\pi(t)}{4\pi'(x_i)\,\pi'(x_k)(s - x_i)(s - x_k)(t - x_i)(t - x_k)}.$$

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* Klein, Math. Ann., Bd. 32, p. 379.

But by decomposition into partial fractions

$$\sum_k \frac{F(x_i, x_k)}{(s - x_k)(t - x_k)\pi'(x_k)} = \frac{1}{t-s}\left[\frac{F(x_i, s)}{\pi(s)} - \frac{F(x_i, t)}{\pi(t)}\right],$$

likewise
$$\sum_i \frac{F(x_i, s)}{\pi'(x_i)(s - x_i)(t - x_i)} = \frac{1}{t-s}\left[\frac{F(s, s)}{\pi(s)} - \frac{F(s, t)}{\pi(t)}\right],$$

and remembering that

$$F(s, s) = R(s), \quad F(t, t) = R(t),$$

we finally obtain the

Theorem: If

$$w_a = w_a^{x_1 a_1} + w_a^{x_2 a_2} + \ldots . w_a^{x_{\rho+1} a_{2\rho+1}},$$

then

$$\left.\begin{aligned}\wp_0(w_1, w_2, \ldots . w_\rho;\, s,\, t) = \pi(s)\pi(t)\left[\sum_i \frac{y_i}{2(s - x_i)(t - x_i)\pi'(x_i)}\right]^2 \\ + \frac{F(s, t)}{2(t-s)^2} - \frac{R(s)\pi(t)}{4(t-s)^2\pi(s)} - \frac{R(t)\pi(s)}{4(s-t)^2\pi(t)}.\end{aligned}\right\} \quad (B'')$$

The special case $s = t$: If we wish to apply (B″) to the case $s = t$ we must first perform the division by $(t - s)^2$; to this effect, introduce homogeneous variables, multiply by $s_2^{\rho-1} t_2^{\rho-1}$ and use symbolical notation

$$R(t) = a_t^{2\rho+2}, \quad F(s, t) = a_t^{\rho+1} a_s^{\rho+1}, \quad \pi(t) = \pi_t^{\rho+1} = p_t^{\rho+1}.$$

The last three terms of (B″) become

$$-\frac{(a_s^{\rho+1}\pi_t^{\rho+1} - a_t^{\rho+1}\pi_s^{\rho+1})(a_s^{\rho+1}p_t^{\rho+1} - a_t^{\rho+1}p_s^{\rho+1}}{4(st)^2\pi(t)\pi(s)};$$

now the division can be immediately performed, and if we put afterwards $s = t$, we get

$$\wp_0(w_1, \ldots . w_\rho;\, t,\, t) = \pi(t)^2\left[\sum_i \frac{y_i}{2(tx_i)^2 \prod_k{}'(x_i x_k)}\right]^2 - \frac{(\rho+1)^2(a\pi)(ap)\, a_t^{2\rho}\pi_t^\rho p_t^\rho}{4\pi(t)^2}. \quad (36)$$

§8.—*Generalization of* (c) *and* (d).

Giving special values either to the upper limits $x_1, x_2, \ldots . x_{\rho+1}$ or to the auxiliary variables s and t, we obtain a series of consequences of our last theorem which are extensions of well-known properties of the elliptic function $\wp u$.

In the first place, put in (B′) $t = x_{\rho+1}$ and observe that

$$\left[\frac{\pi(t)}{(t-x_i)(t-x_k)}\right]_{t=x_{\rho+1}} = \begin{cases} 0 \text{ if } i \text{ and } k \ddagger \rho+1, \\ \dfrac{\pi'(x_{\rho+1})}{x_{\rho+1}-x_i} \text{ if } k = \rho+1; \end{cases}$$

(B′) reduces to

$$\wp_0(w_1, w_2, \ldots\ldots w_\rho;\ s, x_{\rho+1}) = \sum_\alpha \frac{\Phi(s)}{(s-x_\alpha)\Phi'(x_\alpha)} \frac{F(x_\alpha, x_{\rho+1}) - y_\alpha y_{\rho+1}}{2(x_\alpha - x_{\rho+1})^2},$$

where, as in §4,

$$\Phi(s) = (s-x_1)(s-x_2)\ldots\ldots(s-x_\rho). \tag{27}$$

Now give $x_{\rho+1}$ the special value $x_{\rho+1} = a_{2\rho+1}$ and denote, as in §4,

$$u_\alpha = w_\alpha^{x_1 a_1} + w_\alpha^{x_2 a_2} + \ldots\ldots + w_\alpha^{x_\rho a_{2\rho-1}}; \tag{28}$$

then
$$\wp_0(u_1, u_2, \ldots\ldots u_\rho;\ s, a_{2\rho+1}) = \sum_\alpha \frac{\Phi(s)}{\Phi'(x_\alpha)(s-x_\alpha)} \frac{F(x_\alpha, a_{2\rho+1})}{2(x_\alpha - a_{2\rho+1})^2}.$$

But a being any branchpoint, we have

$$F(a, a) = R(a) = 0,$$
$$\left(\frac{dF(x, a)}{dx}\right)_{x=a} = \frac{1}{2} R'(a),$$

therefore
$$\frac{F(x, a) - \frac{1}{2} R'(a)(x-a)}{2(x-a)^2} = G(x)$$

is an *integral* function of x of degree $\rho - 1$, hence

$$\sum_\alpha \frac{\Phi(s)\, G(x_\alpha)}{\Phi'(x_\alpha)(s-x_\alpha)} = G(s).$$

On the other hand, decomposition into partial fractions of $\dfrac{1}{\Phi(s)(s-a)}$ gives

$$\sum_\alpha \frac{\Phi(s)}{\Phi'(x_\alpha)(s-x_\alpha)(x_\alpha-a)} = \frac{1}{s-a} - \frac{\Phi(s)}{\Phi(a)(s-a)}.$$

These transformations lead to the following

Theorem: If $$u_\alpha = w_\alpha^{x_1 a_1} + w_\alpha^{x_2 a_2} + \ldots\ldots w_\alpha^{x_\rho a_{2\rho-1}},$$

then $$\wp_0(u_1, u_2, \ldots\ldots u_\rho;\ s, a_{2\rho+1}) \frac{F(s, a_{2\rho+1})}{2(s-a_{2\rho+1})^2} - \frac{1}{4} \frac{R'(a_{2\rho+1})\Phi(s)}{\Phi(a_{2\rho+1})(s-a_{2\rho+1})}. \tag{C}$$

Special case $\rho = 1$, $g_1(x) = 1$:

$$\wp_0(w_1^{x_1 a_1}) = \frac{F(s,\, a_3)}{2(s-a_3)^2} - \frac{1}{4}\,\frac{R'(a_3)}{s-a_3} + \frac{1}{4}\,\frac{R'(a_3)}{x-a_3}.$$

But
$$F(s,\, a_3) = \frac{1}{2} R'(a_3)(s-a_3) + \frac{1}{12} R''(a_3)(s-a_3)^2,$$

hence
$$\wp_0(w_1^{x_1 a_1}) = \frac{1}{24} R''(a_3) + \frac{1}{4}\,\frac{R'(a_3)}{x_1-a_3} = \wp(w_1^{x_1 a_3}),$$

(C) is therefore indeed an extension of the theorem (c) of the introduction.

If $a_{2\rho+1}$ is made infinitely great, (C) changes into a formula given by Wiltheiss. In fact, choose $g_\alpha(x) = \frac{x^{\alpha-1}}{2}$ and put $a_{2\rho+1} = \frac{1}{\varepsilon}$; multiply (C) by $\varepsilon^{\rho-1}$ and pass to the limit $\varepsilon = 0$, while at the same time $\lim \frac{A_0}{\varepsilon}$ remains finite, say

$$\lim \frac{A_0}{\varepsilon} = (2\rho+2)\, A_1',$$

so that
$$\lim R(x) = \binom{2\rho+2}{1} A_1' x^{2\rho+1} + \binom{2\rho+2}{2} A_2' x^{2\rho} + \ldots . A_{2\rho+2}'.$$

The result of the limiting process is

$$\sum_\alpha \frac{\partial^2 \log \Theta}{\partial u_\alpha\, \partial u_\rho} s^{\alpha-1} = -\,2F(s) + (2\rho+2)\, A_1' \Phi(s), \tag{C$'$}$$

where
$$F(s) = \binom{\rho+1}{1} A_1' s^\rho + \binom{\rho+1}{2} A_2' s^{\rho-1} + \ldots . + A_{\rho+1}.$$

This is the formula given by Wiltheiss, Math. Ann., Bd. 31, p. 414, (8).

Theorem (C) contains a *solution of the inversion problem:*

$$u_\alpha = w_\alpha^{x_1 a_1} + w_\alpha^{x_2 a_2} + \ldots . + w_\alpha^{x_\rho a_{2\rho-1}}.$$

For the equation of degree ρ, $\Phi(s) = 0$, which determines the ρ values of $x_1, x_2, \ldots . x_\rho$, is, according to (C),

$$(s - a_{2\rho+1})\, \wp_0(u_1, u_2, \ldots . u_\rho;\; s,\, a_{2\rho+1}) - \frac{F(s,\, a_{2\rho+1})}{2(s-a_{2\rho+1})} = 0. \tag{C$''$}$$

(Remember that $F(s,\, a_{2\rho+1})$ is divisible by $s - a_{2\rho+1}$).

The values $x_1, x_2, \ldots . x_\rho$ being found, for every x_δ the corresponding value y_δ is given by the equation

$$\sum_{\alpha,\beta,\gamma} \frac{\partial^3 \log \sigma_0}{\partial u_\alpha\, \partial u_\beta\, \partial u_\gamma} g_\alpha(a_{2\rho+1})\, g_\beta(a_{2\rho+1})\, g_\gamma(x_\delta) = \frac{1}{4}\,\frac{R'(a_{2\rho+1})}{(a_{2\rho+1}-x_\delta)^2}\, y_\delta, \tag{D}$$

an extension of (d) which is derived from (C) by differentiating with respect to x_2 and putting afterwards $s = a_{2\rho+1}$.

§9.—*Generalization of* (e).

Put in (B″) the $\rho + 1$ upper limits $x_1, x_2, \ldots . x_{\rho+1}$ equal to $\rho + 1$ distinct branchpoints, say

$$b_1, b_2, \ldots . b_{\rho+1},$$

and let

$$\left.\begin{aligned} \phi(x) &= (x - b_1)(x - b_2) \ldots . (x - b_{\rho+1}), \\ \psi(x) &= \frac{R(x)}{\phi(x)}, \end{aligned}\right\} \tag{37}$$

and

$$\omega_a = w_a^{b_1 a_1} + w_a^{b_2 a_2} + \ldots . w_a^{b_{\rho+1} a_{2\rho+1}}, \tag{38}$$

(B″) becomes

$$\wp_0(\omega_1, \omega_2, \ldots . \omega_\rho; s, t) = \frac{2F(s, t) - \phi(s)\psi(t) - \phi(t)\psi(s)}{4(t - s)^2}. \tag{E}$$

Introduce homogeneous variables* and use symbolical notation,

$$R(t) = a_t^{2\rho+2}, \ \phi(t) = \phi_t^{\rho+1}, \ \psi(t) = \psi_t^{\rho+1},$$

then

$$F(s, t) = a_s^{\rho+1} a_t^{\rho+1} = \frac{1}{\binom{2\rho+2}{\rho+1}} \Big\{ \phi_s^{\rho+1}\psi_t^{\rho+1} + \binom{\rho+1}{1}^2 \phi_s^\rho \psi_t^\rho \phi_t \psi_s$$
$$+ \binom{\rho+1}{2}^2 \phi_s^{\rho-1}\psi_t^{\rho-1}\phi_t^2\psi_s^2 + \ldots . + \phi_t^{\rho+1}\psi_s^{\rho+1} \Big\},$$

and if we make use of the identity

$$1 + \binom{\rho+1}{1}^2 + \binom{\rho+1}{2}^2 + \ldots . + \binom{\rho+1}{1}^2 + 1 = \binom{2\rho+2}{\rho+1}$$

we obtain, after a few simplifications, the *result:*

$$\wp_0(\omega_1, \omega_2, \ldots . \omega_\rho; s, t) = - \frac{(\phi\psi)^2 \sum_a \binom{\rho+1}{a}^2 S_{a-1} S_{\rho-a}}{4\binom{2\rho+2}{\rho+1}}, \tag{E′}$$

where $S_i = \phi_s^i \psi_t^i + \phi_s^{i-1}\phi_t \psi_s \psi_t^{i-1} + \ldots . + \phi_s \phi_t^{i-1} \psi_s^{i-1} \psi_t + \phi_t^i \psi_s^i$

and $S_0 = 1$.

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* See the agreement concerning notation at the end of §1.

For $\rho = 1$, (E′) gives

$$\wp_0(\omega_1) = -\frac{1}{6}(\phi\psi)^2,$$

where.
$$\omega_1 = w_1^{b_1a_1} + w_1^{b_2a_2};$$

but
$$\wp_0(\omega_1) = \wp_0(w_1^{b_1b_2} + w_1^{a_1a_2}) = \wp(w_1^{b_1b_2});$$

hence $\wp\left(\int_{b_2}^{b_1}\frac{dx}{y}\right) = -\frac{1}{6}(\phi\psi)^2$, which is theorem (e) of the introduction.

For $\rho = 2$, (E′) gives

$$\wp_0(\omega_1, \omega_2; s, t) = -\frac{9}{40}\{(\phi\psi)^2\phi_s\psi_t + (\phi\psi)^2\phi_t\psi_s\}. \tag{39}$$

If the auxiliary variables s and t are replaced by two of the branchpoints b, say b_i, b_k, (E) becomes

$$\wp_0(\omega_1, \omega_2, \dots\dots \omega_\rho; b_i, b_k) = \frac{F(b_i, b_k)}{2(b_i - b_k)^2}. \tag{40}$$

§10.—*Generalization of* $\wp u - e_\lambda = \frac{\sigma_\lambda^2 u}{\sigma^2 u}$.

A first extension of this theorem is immediately derived from (C) by replacing s by a branchpoint different from $a_{2\rho+1}$, say

$$s = a_\lambda, \quad \lambda \neq 2\rho + 1.^*$$

We obtain

$$\wp(u_1, \dots\dots u_\rho; a_\lambda, a_{2\rho+1}) - \frac{F(a_\lambda, a_{2\rho+1})}{2(a_\lambda - a_{2\rho+1})^2} = -\frac{1}{4}\frac{R'(a_{2\rho+1})\Phi(a_\lambda)}{(a_\lambda - a_{2\rho+1})\Phi(a_{2\rho+1})}. \tag{41}$$

But if we use Weierstrass'† index notation

$$\frac{\Phi(a_\lambda)}{\Phi(a_{2\rho+1})} = \frac{\sqrt{(-1)^\lambda R'(a_\lambda)}}{\sqrt{-R'(a_{2\rho+1})}}\,\frac{\Theta^2(u_1, \dots\dots u_\rho)\lambda}{\Theta^2(u_1, \dots\dots u_\rho)}; \tag{42}$$

hence if we write
$$\frac{F(a_i, a_k)}{2(a_i - a_k)^2} = e_{ik}, \tag{43}$$

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* For $\lambda = 2\rho + 1$ we get

$$\wp_0(u_1, \dots\dots u_\rho; a_{2\rho+1}, a_{2\rho+1}) = \frac{\rho}{8(2\rho+1)}R''(a_{2\rho+1}) - \tfrac{1}{4}R'(a_{2\rho+1})\frac{\Phi'(a_{2\rho+1})}{\Phi(a_{2\rho+1})}.$$

† Weierstrass, Lectures on Hyperelliptic Functions, Solution of the inversion problem.

we obtain the *result:*

$$\wp_0(u_1, \ldots\ldots u_\rho;\, a_\lambda,\, a_{2\rho+1}) = e_{\lambda, 2\rho+1}$$
$$= \frac{1}{4}\,\frac{\sqrt{(-1)^{\lambda+1} R'(a_\lambda)\, R'(a_{2\rho+1})}}{a_\lambda - a_{2\rho+1}}\;\frac{\Theta^2(u_1, \ldots\ldots u_\rho)_\lambda}{\Theta^2(u_1, \ldots\ldots u_\rho)}. \qquad \text{(F)}$$

For $\rho = 2$ and $a_{2\rho+1} = \infty$, this result was first proved by Brioschi, Rend. della Acc. dei Lincei, 1886_1, p. 199; the general formula (F) was given by Wiltheiss, Math. Ann., Bd. 31, p. 417.

An analogous formula for Θ-functions with two indices is obtained by putting in (B″)

$$x_{\rho+1} = a_{2\rho+1}, \quad s = a_\lambda, \quad t = a_\mu.$$

(B″) becomes

$$\wp_0(u_1, \ldots\ldots u_\rho;\, a_\lambda,\, a_\mu) - \frac{F(a_\lambda, a_\mu)}{2(a_\lambda - a_\mu)^2}$$
$$= (a_\lambda - a_{2\rho+1})(a_\mu - a_{2\rho+1})\,\Phi(a_\lambda)\Phi(a_\mu)\left\{\sum_\alpha \frac{y_\alpha}{2(x_\alpha - a_\lambda)(x_\alpha - a_\mu)(x_\alpha - a_{2\rho+1})\Phi'(x_\alpha)}\right\}^2. \qquad (44)$$

But according to Weierstrass,

$$\frac{(a_{2\rho+1} - a_\lambda)(a_\mu - a_\lambda)(a_\mu - a_{2\rho+1})\,\Phi^2(a_{2\rho+1})}{R'(a_{2\rho+1})}\left\{\sum_\alpha \frac{y_\alpha}{(x_\alpha - a_\lambda)(x_\alpha - a_\mu)(x_\alpha - a_{2\rho+1})\Phi'(x_\alpha)}\right\}^2$$
$$= \pm\frac{\Theta^2(u_1, \ldots\ldots u_\rho)_{\lambda\mu}\,\Theta^2(u_1, \ldots\ldots u_\rho)}{\Theta^2(u_1, \ldots\ldots u_\rho)_\lambda\,\Theta^2(u_1, \ldots\ldots u_\rho)_\mu}. \qquad (45)$$

Hence we obtain the *result:*

$$\wp_0(u_1, u_2, \ldots\ldots u_\rho;\, a_\lambda a_\mu) - e_{\lambda\mu}$$
$$= \frac{1}{4}\,\frac{\sqrt{(-1)^{\lambda} R'(a_\lambda)}\sqrt{(-1)^{\mu} R'(a_\mu)}}{a_\lambda - a_\mu}\;\frac{\Theta^2(u_1, u_2, \ldots\ldots u_\rho)_{\lambda\mu}}{\Theta^2(u_1, u_2, \ldots\ldots u_\rho)}. \qquad \text{(G)}$$

This agrees exactly with the result given by Brioschi l. c. for $\rho = 2$, $a_{2\rho+1} = \infty$, if we make use of the identity

$$\frac{F(a_\lambda, a_\mu)}{2(a_\lambda - a_\mu)^2} = \frac{1}{4}\,\frac{R'(a_\lambda)}{a_\mu - a_\lambda} - \sum_\alpha g_{\rho+\alpha}(a_\lambda)\, g_\alpha(a_\mu) = \frac{1}{4}\,\frac{R'(a_\mu)}{a_\lambda - a_\mu} - \sum_\alpha g_{\rho+\alpha}(a_\mu)\, g_\alpha(a_\lambda),$$

which follows from (5) for $\xi = a_\lambda$, $x = a_\mu$.

The symmetry with respect to the branchpoints is better seen, and at the same time the irrational coefficients are avoided, if Klein's σ-*functions and alge-*

braic characteristics are introduced instead of Weierstrass' Θ-functions and index notation. This may be done either by means of the relations which connect the Θ- and σ-functions (see Klein, Math. Ann., Bd. 32, p. 376, and Schröder, Ueber den Zusammenhang der hyperelliptischen σ- und ϑ-Functionen, Diss. Göttingen, 1890), or else by using Klein's theorem* (15). The *result* is as follows:

Let as before

$$\left.\begin{aligned}\phi_0(x) &= \text{const.}\,(x-a_1)(x-a_3)\,\ldots\,(x-a_{2\rho+1}),\\ \psi_0(x) &= \text{const.}\,(x-a_0)(x-a_2)\,\ldots\,(x-a_{2\rho})\end{aligned}\right\} \qquad (8)$$

and

$$\sigma_0 = \sigma_{\phi_0\psi_0}.$$

Two essentially different cases have to be distinguished.

I. *Case*: a_λ, a_μ belong the one to ϕ_0, the other to ψ_0, say $\lambda = 2i-1$, $\mu = 2k$. Put

$$\left.\begin{aligned}\phi(x) &= \frac{x-a_{2k}}{x-a_{2i-1}}\,\phi_0(x),\\ \psi(x) &= \frac{x-a_{2i-1}}{x-a_{2k}}\,\psi_0(x),\end{aligned}\right\} \qquad (46)$$

then

$$\wp_0(u_1, u_2, \ldots u_\rho;\, a_{2i-1}, a_{2k}) - e_{2i-1,\,2k} = \frac{1}{4}\,\phi'(a_{2k})\,\psi'(a_{2i-1})\,\frac{\sigma^2_{\phi\psi}(u_1, u_2, \ldots u_\rho)}{\sigma^2_0(u_1, u_2, \ldots u_\rho)}. \quad (\mathrm{H})$$

II. *Case*: a_λ, a_μ belong both to the same of the two factors ϕ_0, ψ_0, say for instance to ϕ_0, $\lambda = 2i-1$, $\mu = 2k-1$. Put

$$\begin{aligned}p(x) &= \frac{\phi_0(x)}{(x-a_{2i-1})(x-a_{2k-1})}, \qquad (47)\\ \chi(x) &= \psi_0(x)(x-a_{2i-1})(x-a_{2k-1}),\end{aligned}$$

then

$$\begin{aligned}&\wp_0(u_1, u_2, \ldots u_\rho;\, a_{2i-1}, a_{2k-1}) - e_{2i-1,\,2k-1}\\ &\qquad = \frac{1}{16}\,\psi_0(a_{2i-1})\,\psi_0(a_{2k-1})\,\frac{\sigma^2_{p\chi}(u_1, u_2, \ldots u_\rho)}{\sigma^2_0(u_1, u_2, \ldots u_\rho)}. \quad (\mathrm{I})\end{aligned}$$

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* For the value of the constant factor $c_\nu^{(\mu)}$ see Burkhardt, Math. Ann., Bd. 32, p. 418.

Verification for $\rho = 1$: Choose $g_1(x) = 1$ and the constant factor in $\phi_0(x)$ equal 1; then

$$\phi_0(x) = (x - a_1)(x - a_3), \quad \psi_0(x) = A_0(x - a_0)(x - a_2),$$
$$p(x) = 1, \qquad \chi(x) = R(x).$$

Further choose, in order to fix the ideas, in (46),

$$a_{2i-1} = a_3, \quad a_{2k} = a_0,$$

so that $$\phi(x) = (x - a_0)(x - a_1), \quad \psi(x) = A_0(x - a_2)(x - a_3);$$

(H) and (I) become

$$\wp_0(u) - e_{03} = \frac{1}{4} A_0(a_0 - a_1)(a_3 - a_2)\frac{\sigma^2_{\phi\psi}(u)}{\sigma^2_0(u)}, \tag{h}$$

$$\wp_0(u) - e_{13} = \frac{1}{16} A_0^2(a_1 - a_0)(a_1 - a_2)(a_3 - a_0)(a_3 - a_2)\frac{\sigma^2(u)}{\sigma^2_0(u)}. \tag{i}$$

But if we express $F(a_i, a_k)$ in terms of the roots we obtain*

$$e_{ik} = \frac{F(a_i, a_k)}{2(a_i - a_k)^2} = -\frac{A_0}{12}\left[(a_i - a_l)(a_k - a_m) + (a_i - a_m)(a_k - a_l)\right],$$

i, k, l, m denoting the four numbers 0, 1, 2, 3 taken in any order; therefore

$$e_{ik} = e_{lm}$$

and $$e_{ik} - e_{il} = -\frac{A_0}{4}(a_i - a_m)(a_k - a_l).$$

Hence if we designate

$$e_{13} = e_{02} = e_\lambda; \quad e_{10} = e_{23} = e_\mu; \quad e_{12} = e_{03} = e_\nu,$$
$$\omega_\lambda = \int_{a_1}^{a_3} \frac{dx}{y},$$

and accordingly $$\sigma_0 = \sigma_\lambda, \quad \sigma_{\phi\psi} = \sigma_\mu,$$

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[*] Compare Weierstrass, Lectures on Elliptic Functions, and Klein, Elliptische Modulfunctionen, I, p. 17.

(h) and (i) become

$$\wp(u+\omega_\lambda)-e_\nu=(e_\lambda-e_\nu)\frac{\sigma_\mu^2(u)}{\sigma_\lambda^2(u)}, \tag{h'}$$

$$\wp(u+\omega_\lambda)-e_\lambda=(e_\lambda-e_\mu)(e_\lambda-e_\nu)\frac{\sigma^2(u)}{\sigma_\lambda^2(u)}, \tag{i'}$$

and if we apply the formula*

$$\wp(u+\omega_\lambda)-e_\lambda=\frac{(e_\lambda-e_\mu)(e_\lambda-e_\nu)}{\wp u-e_\lambda}, \tag{48}$$

we obtain the well-known equations

$$\frac{\wp(u)-e_\mu}{\wp(u)-e_\lambda}=\frac{\sigma_\mu^2(u)}{\sigma_\lambda^2(u)}, \tag{h''}$$

$$\frac{1}{\wp(u)-e_\lambda}=\frac{\sigma^2(u)}{\sigma_\lambda^2(u)}. \tag{i''}$$

An *extension of* (48) may be deduced from (H) as follows: Interchange the two branchpoints a_{2i-1} and a_{2k}, the result is

$$\wp_{\phi\psi}(u_1,\ldots\ldots u_\rho;\ a_{2k},a_{2i-1})-e_{2i-1,\,2k}=\frac{1}{4}\,\phi_0'(a_{2i-1})\,\psi_0'(a_{2k})\,\frac{\sigma_0^2(u_1,\ldots\ldots u_\rho)}{\sigma_{\phi\psi}^2(u_1,\ldots\ldots u_\rho)}. \tag{49}$$

Multiply (H) and (49) and observe that (31),

$$\wp_{\phi\psi}(u_1,\ldots\ldots u_\rho;\ s,t)=\wp_0(u_1+\varpi_1,\ldots\ldots u_\rho+\varpi_\rho;\ s,t) \tag{50}$$

if

$$\varpi_a=\int_{a_{2i-1}}^{a_{2k}}\frac{g_a(x)\,dx}{y}, \tag{51}$$

therefore

$$\wp_0(u_1+\varpi_1,\ldots\ldots u_\rho+\varpi_\rho;\ a_{2i-1},a_{2k})-e_{2i-1,\,2k}$$
$$=\frac{1}{16}\,\frac{\phi'(a_{2k})\,\psi'(a_{2i-1})\,\phi_0'(a_{2i-1})\,\psi_0'(a_{2k})}{\wp_0(u_1,\ldots\ldots u_\rho;\ a_{2i-1},a_{2k})}. \tag{K}$$

In concluding, I may remark that most of our developments—in particular the fundamental equations (A), (B), (B″)—are independent of the special assumption which we made concerning the commutative integral of the third kind. We might just as well have chosen, instead of Klein's integral Q, the *most general commutative integral of the third kind,* and accordingly, instead of Klein's σ-functions, the most general σ-functions. We would only have to replace in all our formulæ the $\rho+1$st polar $F(x,\xi)$ by the most general integral symmetric function of x and ξ, $F(x, \xi)$ of degree $\rho + 1$ with respect to each of the two variables satisfying the two conditions*

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*Schwarz's Formelsammlung, Art. 19, (5).

$$F(\xi, \xi) = R(\xi),$$
$$\left(\frac{\partial F(x, \xi)}{\partial x}\right)_{x=\xi} = \frac{1}{2} R'(\xi).$$

Our results would thus gain in generality, but they would lose their covariant character.

FREIBURG i. B., *August*, 1894.

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* Weierstrass, Lectures on Hyperelliptic Functions; Klein, Math. Ann., Bd. 27, p. 441, and Lectures on Hyperelliptic Functions, 1887-88.

Sur la definition de la limite d'une fonction. Exercice de logique mathématique.

PAR G. PEANO à *Turin.*

Selon la définition de la limite, aujourd'hui adoptée dans tous les traités, toute fonction a une limite seule, ou n'a pas de limite; une variable ne peut tendre à la fois vers deux limites différentes (voir p. ex. Jordan, Cours d'Analyse, 1892, pag. 9).

Mais, en faisant un petit changement dans la définition commune, on obtient une nouvelle définition de la limite; alors toute fonction a des limites finies ou infinies; si la limite est unique, on retrouve, comme cas particulier, les propositions communes.

J'ai donné cette nouvelle définition dans la Rivista di Matematica, 1892, p. 77, et je l'ai adoptée dans mes Lezioni di Analisi infinitesimale (1893); mais on rencontre la même idée dans des publications antérieures. M. Giudice dans un intéressant article (Sulle successioni, Rendiconti di Palermo, t. V, a. 1891, pag. 280), énonce les propriétés plus importantes des suites, et les décompose en suites convergentes. P. du Bois-Reymond a introduit les *Unbestimmtheitsgrenzen* qui sont la plus petite et la plus grande des limites que nous allons définir. Mais on les trouve dans cette proposition de Abel (Œuvres, II, pag. 199):

"Pour qu'une série Σu_n soit convergente, il faut que *la plus petite des limites* de nu_n soit zéro."

Cette idée plus générale de la limite est clairement énoncée par Cauchy; on lit en effet dans son Cours d'Analyse algébrique, 1821, p. 13:

"Quelquefois .... une expression converge à-la-fois vers plusieurs limites différentes les unes des autres,"

et à pag. 14 il trouve que les valeurs limites de $\sin\frac{1}{x}$, pour $x=0$, constituent l'intervalle de -1 à $+1$. Les auteurs qui ont suivi Cauchy, en cherchant de préciser sa définition un peu vague, se sont mis dans un cas particulier.

L'objet de cette Note est de donner cette définition générale de la limite, et de démontrer ses principales propriétés. Nous ferons usage, dans cette étude, de la Logique mathématique. Cette science s'est rapidement développée de nos jours, et on l'a appliquée dans plusieurs travaux. Nous aurons à citer le Formulaire de Mathématique (abrégé en Form.), que publie maintenant une société de professeurs, dans la Rivista di Matematica.

L'introduction à ce formulaire explique les notations; la première partie contient toutes les propositions ou règles de logique qu'on a jusqu'à présent rencontrées. Pour notre travail, nous aurons aussi occasion de citer la partie V, qui regarde la théorie des ensembles de nombres, des limites supérieures et inférieures, des ensembles dérivés, etc.

M. Burali-Forti vient de publier son traité "Logica matematica" (Milano, Hœpli, 1894), livre d'un grand avantage pour ceux qui vourront se mettre au courant de cette science. On trouvera des expositions plus ou moins rapides des notations toutes les fois que j'en ai fait usage; notamment dans l'article publié dans les Mathematische Annalen, t. XXXVII.

Mais, pour notre but, il suffit une connaissance sommaire des notations; c'est-à-dire des signes de logique $\frown$, $\smile$, $\sim$, Λ, Ɔ, ε, $=$, K, des points, du signe de fonction f, et des signes de mathématique Q, q, N, l', l_1, et quelques autres qu'on rencontrera dans la suite. Les règles seront expliquées dans chaque cas, en renvoyant au Formulaire pour des plus amples explications; ainsi cet article sera un exercice de Logique mathématique.

Soit fx une fonction réelle de la variable réelle x. Cette variable x peut tendre vers une valeur finie a, ou vers $\pm\infty$; mais par le changement de variable $x = a + \frac{1}{x'}$, le premier cas est réduit au deuxième. Si x tend vers $-\infty$, en posant $x = -x'$, x' tend vers $+\infty$. Donc, sans ôter à la généralité de la question, on peut supposer que la variable indépendante tende vers $+\infty$.

La fonction fx sera définie pour les valeurs de la variable appartenant à un ensemble u. Si cet ensemble coincide avec l'ensemble des nombres réels, c'est-à-dire si $u = \mathrm{q}$, la fonction sera définie pour toutes les valeurs réelles de la variable, comme x^2, $\sin x, \ldots.$; si $u = \mathrm{Q}$, la fonction sera donnée pour les valeurs positives, comme $\log x$, $\sqrt{x}, \ldots.$

Si u est l'ensemble des nombres entiers positifs, $u = \mathrm{N}$, ont signification les symboles $f1, f2, \ldots.$ qui forment une *suite*. Mais la question générale de la limite d'une fonction n'est pas réductible au cas particulier de la limite d'une suite, sans perdre de sa généralité ; car si u est un ensemble continu, p. ex. $u = \mathrm{q}$, on ne peut pas mettre ses individus en correspondance semblable avec les nombres 1, 2, 3, ... En conséquence les auteurs qui, en définissant la limite, parlent d'une suite $x_1, x_2, \ldots\, x_n, \ldots$ se placent dans un cas trop particulier, duquel on ne peut tirer la définition de la limite d'une fonction, dont on fera usage dans l'analyse infinitésimale.

Afin que la variable x puisse tendre vers $+\infty$, il faut que la classe u de ses valeurs contienne des nombres aussi grands que l'on veut ; c'est-à-dire que la classe u soit illimitée supérieurement, ou la limite supérieure des u soit ∞. En répétant ces conditions, nous pouvons donner la définition de la limite :

Définition.—"Soit u une classe de quantités $[u \varepsilon \mathrm{Kq}]$, dont la limite supérieure est infinie $[\mathrm{l}'u = \infty]$, et soit f le signe d'une fonction réelle définie dans la classe u $[f \varepsilon \mathrm{q}fu]$. Étant y une quantité finie $[y \varepsilon \mathrm{q}]$, nous dirons que y est une valeur limite de fx, lorsque x, en variant dans la classe u, tend à l'∞, [et nous écrirons $y \varepsilon \lim_{x,u,\infty} fx$], lorsque, étant donné un nombre positif arbitrairement petit h $[h \varepsilon \mathrm{Q}]$, quelque soit le nombre a, aussi grand que l'on veut $[a \varepsilon \mathrm{q}]$, il y a toujours de valeurs de x, contenues dans l'ensemble u $[x \varepsilon u]$, et plus grandes que a $[x > a]$, qui rendent la différence $fx - y$, en valeur absolue, plus petite que h $[\mathrm{mod}\,(fx - y) < h]$."

En liant ces conditions par les notations de logique, cette définition s'énonce :

1. $u \varepsilon \mathrm{Kq} \,.\, \mathrm{l}'u = \infty \,.\, f \varepsilon \mathrm{q}fu \,.\, y \varepsilon \mathrm{q} \,.\!\supset ::$
$y \varepsilon \lim_{x,u,\infty} fx \,.\!=\therefore h \varepsilon \mathrm{Q} \,.\, a \varepsilon \mathrm{q} \,.\!\supset_{h,a} : x \varepsilon u \,.\, x > a \,.\, \mathrm{mod}\,(fx - y) < h \,.\, \sim =_x \Lambda.$ Def.

Nous ferons quelques remarques sur cette définition. Nous avons transformé les mots "étant donné un nombre positif, arbitrairement petit h" en "$h \varepsilon \mathrm{Q}$," car les mots "arbitrairement petit" sont un pléonasme. Analoguement pour a.

Ce qui précède le signe $\supset ::$ est l'hypothèse de la définition ; elle explique la signification des lettres u, f, et y. Ce qui suit le signe de déduction est l'égalité qui constitue la définition. Son premier membre, qui est en avant du signe $= \therefore$ est l'expression qu'on définit ; le deuxième en indique la valeur.

On dit qu'une lettre variable x est apparente dans une expression qui la contient, si cette expression en est indépendante. Ainsi dans $\int_a^b fx\,\mathrm{d}x$, la lettre x est apparente, car $\int_a^b fx\,\mathrm{d}x = \int_a^b fz\,\mathrm{d}z$. L'expression que nous avons défini,

$y \varepsilon \lim_{x,u,\infty} fx$, contient la lettre apparente x; si l'on ne veut pas de lettre apparente, il suffit de changer de notation; on pourrait écrire p. ex. $y \varepsilon \lim f(\infty, u)$, ou $y \varepsilon f(\infty, u)$, ou $y \varepsilon f_u \infty$, etc. Mais la notation adoptée est la plus commode.

L'expression définie depend effectivement des seules lettres variables y, u, f contenues dans l'hypothèse, et dans le second membre de la définition. Ce second membre est une déduction, dont la thèse est une inégalité logique. Il contient aussi les lettres h, a, x; mais elles sont des lettres apparentes; car la thèse, où figure la lettre x, a au signe $=$ l'indice x; donc cette thèse est indépendante de x. Et les lettres h et a sont des indices au signe de déduction; donc le second membre est aussi indépendant des lettres h et a (voir Intr. au Form., §14). Les deux membres de l'égalité contiennent les mêmes lettres.

D'ordinaire on indique la limite cherchée par $\lim_{x=\infty} fx$, en sousentendant la classe u des valeurs de x. Dans ce qui suit, pour abréger, nous écrirons simplement lim au lieu de $\lim_{x,u,\infty}$; en revenant, lorsqu'il faut, à la notation complète, qui ne produit jamais des ambiguités.

Nous commençons par transformer cette définition. Comme nous venons de dire, elle a la forme

$$(\alpha) \qquad \text{Hypoth.} \supset . \ 1^{\text{er}} \text{ membre} = 2^{\text{ème}} \text{ membre.}$$

Or les deux membres de l'égalité sont des propositions; et l'égalité entre deux propositions a et b, $a = b$, est identique à l'ensemble des propositions $a \supset b . b \supset a$ (voir Form. I, §1, P3).

Donc en exprimant le signe $=$ par le signe $\supset$, la proposition (α) se transforme en

$$(\beta) \quad \text{Hypoth.} \supset . \ 1^{\text{er}} \text{ membre} \supset 2^{\text{ème}} \text{ membre.} \quad 2^{\text{ème}} \text{ membre} \supset 1^{\text{er}} \text{ membre.}$$

Maintenant, étant a, b, c des propositions, la $a \supset bc$ est équivalente à l'ensemble $a \supset b . a \supset c$ (Form. I, §1, P36). Nous dirons, avec M. Burali, qu'on *compose* les propositions $a \supset b . a \supset c$, lorsqu'on les écrit sous la forme $a \supset bc$; et qu'on *décompose* cette ci, lorsqu'on la réduit à la forme primitive. Décomposons donc la (β), dont la thèse est l'ensemble de deux propositions; on obtient $(\beta) = (\gamma)(\delta)$, où

$$(\gamma) \qquad \text{Hyp.} \supset . \ 1^{\text{er}} \text{ membre} \supset 2^{\text{ème}} \text{ membre.}$$

$$(\delta) \qquad \text{Hyp.} \supset . \ 2^{\text{ème}} \text{ membre} \supset 1^{\text{er}} \text{ membre.}$$

Or, étant a, b, c des propositions, l'expression $a\supset . b\supset c$ (de a on déduit que de b on déduit c) est identique à $ab\supset c$ (Form. I, §1, P39, et Intr. §12, P13). On appelle *importer* l'hypothèse a, la transformation de la première à la seconde forme, et *exporter* l'hypothèse a, la transformation inverse. Importons donc l'Hyp. dans les (γ) et (δ) ; elles se transforment en :

(ε) Hyp. 1$^{\text{er}}$ membre. $\supset$ 2$^{\text{ème}}$ membre.

(ζ) Hyp. 2$^{\text{ème}}$ membre. $\supset$ 1$^{\text{er}}$ membre.

Cette transformation de la (α) dans (ε)(ζ) est tout-à-fait générale ; si a, b, c sont des propositions, la $a\supset . b = c$ est identique à $ab\supset c . ac\supset b$, comme dit la P44 du §1 de la première partie du Form. Mais nous avons préferé de faire les transformations une à la fois. Substituons aux abréviations 1$^{\text{er}}$ membre 2$^{\text{ème}}$ membre, ses valeurs. Écrivons, selons les notations adoptées, HpP1 pour indiquer l'hypothèse de la proposition 1. La P1 se transforme dans l'ensemble des prop. 2 et 3 :

2. $\mathrm{HpP1} .\ y\varepsilon \lim fx . \supset \therefore h\varepsilon \mathrm{Q} . a\varepsilon \mathrm{q} . \supset_{h,\, a} : x\varepsilon u . x > a . \mathrm{mod}\,(fx - y) < h . \sim =_x \Lambda .$

3. $\mathrm{HpP1} \therefore h\varepsilon \mathrm{Q} . a\varepsilon \mathrm{q} . \supset_{h,\, a} : x\varepsilon u . x > a . \mathrm{mod}\,(fx - y) < h . \sim =_x \Lambda \therefore \supset . y\varepsilon \lim fx .$

La P2 a encore la forme $a\supset . b\supset c$; importons encore l'hypothèse ; elle se transforme en :

4. $\mathrm{HpP1} .\ y\varepsilon \lim fx . h\varepsilon \mathrm{Q} . a\varepsilon \mathrm{q} . \supset : x\varepsilon u . x > a . \mathrm{mod}\,(fx - y) < h . \sim =_x \Lambda$

qu'on lit "ayant u, f, y la signification expliquée dans les Hypoth. de la P1, si y est une limite de fx, si h est une quantité positive, et si a est une quantité réelle quelconque, il y a toujours de valeurs x, appartenant à l'ensemble u, plus grands que a, et qui rendent la différence $fx - y$ moindre, en valeur absolue, que h."

Remarquons la loi qui règle les indices dans la transformation de la P2 dans la P4. Dans la P2 le signe de déduction principale, $\supset \therefore$, ne porte pas d'indices ; ils sont sousentendus, et sont toutes les lettres variables dont dépendent les deux membres, u, f, y. Le second signe de déduction $\supset :$ a comme indices les lettres h et a. En important l'hypothèse (Int. §18, P2), on mettra au signe unique de déduction $\supset :$ qui remplace les deux, tous les indices des deux signes ; dans notre cas, u, f, y, h, a ; mais puisque dans la P4 ce signe de déduction indique la déduction principale, on les sousentend tous.

La condition entre u, f, y, h, a,

$$(\alpha) \qquad x \varepsilon u \,.\, x > a \,.\, \mathrm{mod}\,(fx - y) < h \,.\, \sim =_x \Lambda,$$

contient la lettre apparente x. On peut la *éliminer*, c'est-à-dire mettre cette condition dans une forme, où ne figure plus la lettre x. À cet effet, exprimons la relation $x > a$, par le signe ε, sous la forme $x \varepsilon a + \mathrm{Q}$; l'ensemble $x \varepsilon u \,.\, x > a$ devient $x \varepsilon u \,.\, x \varepsilon a + \mathrm{Q}$, ou $x \varepsilon u \frown (a + \mathrm{Q})$; car une formule de Logique (Int. §16, P2) dit que, étant a et b des classes, l'ensemble des propositions $x \varepsilon a \,.\, x \varepsilon b$ est identique à la prop. $x \varepsilon a \frown b$. Au lieu de $\mathrm{mod}\,(fx - y) < h$, on peut écrire $\mathrm{mod}\,(fx - y)\, \varepsilon h - \mathrm{Q}$; et la proposition ($\alpha$) se transforme d'abord en:

$$(\beta) \qquad x \varepsilon u \frown (a + \mathrm{Q}) \,.\, \mathrm{mod}\,(fx - y)\, \varepsilon h - \mathrm{Q} \,.\, \sim =_x \Lambda.$$

Maintenant, étant u une classe, et g un signe de fonction, par gu on entend l'ensemble des valeurs de la fonction gx, lorsque x prend toutes les valeurs dans la classe u; et alors, étant v une autre classe, on a la formule de Logique (Int. §29, P5),

$$x \varepsilon u \,.\, gx \varepsilon v \,.\, \sim =_x \Lambda \,:=.\, gu \frown v \sim = \Lambda,$$

"dire qu'il y a un x, appartenant à la classe u, dont le gx est un v, est identique à dire que les classes gu et v ont des individus communs." Appliquons cette transformation à notre cas. Posons donc $u \frown (a + \mathrm{Q})$ au lieu de u, $\mathrm{mod}\,(fx - y)$ au lieu de gx, et $h - \mathrm{Q}$ au lieu de v. La (β) devient

$$(\gamma) \qquad \mathrm{mod}\,\{f[u \frown (a + \mathrm{Q})] - y\} \frown (h - \mathrm{Q}) \sim = \Lambda$$

et ainsi on a éliminé la x. En substituant dans la P1, on a:

5. $\mathrm{Hp\,P1} \,.\, \supset \therefore y \varepsilon \lim fx \,.\, = : h \varepsilon \mathrm{Q} \,.\, a \varepsilon \mathrm{q} \,.\, \supset_{h,\, a} \,.\, \mathrm{mod}\{f[u \frown (a + \mathrm{Q})] - y\} \frown (h - \mathrm{Q}) \sim = \Lambda.$

On peut présenter sous une autre forme cette élimination. Étant a et b deux quantités, et $a < b$, on désigne par $a \,\overline{}\, b$ l'intervalle de a à b, c'est-à-dire $(a + \mathrm{Q}) \frown (b - \mathrm{Q})$; donc $x \varepsilon a \,\overline{}\, b$ signifie $a < x < b$. Alors la proposition $\mathrm{mod}\,(fx - y) < h$ est équivalente à

$$-h < fx - y < h$$

ou à

$$y - h < fx < y + h$$

et enfin à

$$fx \varepsilon (y - h) \,\overline{}\, (y + h),$$

c'est-à-dire fx appartient à l'intervalle de $y - h$ à $y + h$. Alors la proposition (α) devient

$$(\delta) \qquad x \varepsilon u \frown (a + \mathrm{Q}) \,.\, fx \varepsilon (y - h) \,\overline{}\, (y + h) \,.\, \sim =_x \Lambda.$$

Appliquons maintenant la transformation logique énoncée, en y faisant la substitution

$$\begin{pmatrix} u \frown (a+\mathrm{Q}), & f, & (y-h)\,\overline{\quad}\,(y+h) \\ u, & g, & v \end{pmatrix},$$

c'est-à-dire, en posant au lieu des lettres écrites en bas les expressions supérieures; la (δ) se transforme en:

$$(\varepsilon) \qquad f[u \frown (a+\mathrm{Q})] \frown (y-h)\,\overline{\quad}\,(y+h) \sim = \Lambda,$$

et la définition 1 devient

6. $\mathrm{HpP1}. \supset \therefore y \varepsilon \lim fx . = : h \varepsilon \mathrm{Q} . a \varepsilon \mathrm{q} . \supset_{h,\,a} . f[u \frown (a+\mathrm{Q})] \frown (y-h)\,\overline{\quad}\,(y+h) \sim = \Lambda.$

"Ayant u, f et y la signification expliquée, y est une valeur limite de fx, lorsque, en prenant un nombre positif h et un nombre a, il y a toujours des valeurs de la fonction f, correspondant à des valeurs de la variable de la classe u, et plus grands que a, qui appartiennent à l'intervalle de $y-h$ à $y+h$."

On a donc éliminé la lettre x, et transformé la P1 dans la 5 ou la 6, par la convention sur la signification de fu, lorsque u est une classe. On peut substituer à la condition (α) la (γ) ou la (ε) dans les propositions 2, 3, et 4. En substituant dans la P4 on a:

7. $\mathrm{HpP1}. y \varepsilon \lim fx . h \varepsilon \mathrm{Q} . a \varepsilon \mathrm{q} . \supset . f[u \frown (a+\mathrm{Q})] \frown (y-h)\,\overline{\quad}\,(y+h) \sim = \Lambda.$

Transportons le second membre de la déduction dans le premier, et l'hyp. $y \varepsilon \lim fx$ du premier dans le second, par la règle de logique que $ab \supset c$ est identique à $a \sim c \supset \sim b$ (Form. I, §2, P24); il faut nier les propositions qu'on transporte; la négation du second membre s'obtient en supprimant le signe $\sim$ au devant de l'$=$; et l'on a:

8. $\mathrm{HpP1}. h \varepsilon \mathrm{Q} . a \varepsilon \mathrm{q} . f[u \frown (a+\mathrm{Q})] \frown (y-h)\,\overline{\quad}\,(y+h) = \Lambda . \supset . y \sim \varepsilon \lim fx.$

"En conservant les lettres u, f, y la même signification, si h est un nombre positif, et a une quantité quelconque, et s'il n'y a pas de valeurs de la classe $f[u \frown (a+\mathrm{Q})]$ appartenant à l'intervalle de $y-h$ à $y+h$, alors y n'est pas une limite de fx."

Nous voulons maintenant éliminer la lettre apparente h, qui figure au le second membre de l'égalité dans la définition de la limite. Ce second membre, sous la forme de la P5 est

$$(a) \qquad h \varepsilon \mathrm{Q} . a \varepsilon \mathrm{q} . \supset_{h,\,a} . \bmod \{ f[u \frown (a+\mathrm{Q})] - y \} \frown (h - \mathrm{Q}) \sim = \Lambda.$$

Exportons l'hypothèse $a\varepsilon\mathrm{q}$, selon les règles déjà vues; il se transforme en

$$(\beta)\qquad a\varepsilon\mathrm{q} \,.\, \supset_a : h\varepsilon\mathrm{Q} \,.\, \supset_h . \mathrm{mod}\,\{f[u \frown (a+\mathrm{Q})] - y\} \frown (h - \mathrm{Q}) \sim = \Lambda .$$

"Quelque soit a, alors, quelque soit h, on a etc." Or de la théorie des limites supérieure et inférieure d'un ensemble de nombres, on a (Form. V, §3, P20):

$$u\varepsilon\mathrm{KQ}_0 \,.\, \supset \therefore \mathrm{l}_1 u = 0 \,.=: h\varepsilon\mathrm{Q} \,.\, \supset_h . u \frown (h - \mathrm{Q}) \sim = \Lambda .$$

"Étant u un ensemble de nombres positifs ou nuls, leur limite inférieure sera zéro, lorsque, quelque soit le nombre positif h, il y a toujours des nombres de l'ensemble u plus petits que h." Posons ici $\mathrm{mod}\,\{f[u \frown (a+\mathrm{Q})] - y\}$ au lieu de u; l'hypothèse $u\varepsilon\mathrm{KQ}_0$ est satisfaite, car les modules sont des nombres positifs ou nuls; et on ne l'écrit plus, en vertu du principe de Logique (Form. I, §1, P12) $a . a \supset b . \supset b$: "Si la proposition a est vraie, et de la a on déduit la b, alors la b est aussi vraie." Envertissons les deux membres; nous avons:

$$h\varepsilon\mathrm{Q} \,.\, \supset_h . \mathrm{mod}\,\{f[u \frown (a+\mathrm{Q})] - y\} \frown (h - \mathrm{Q}) \sim = \Lambda :$$
$$= . \mathrm{l}_1\, \mathrm{mod}\,\{f[u \frown (a+\mathrm{Q})] - y\} = 0 .$$

Substituons dans la (β) à la thèse sa valeur ainsi transformée; la prop. 5 se transforme en:

9. $\quad \mathrm{HpP1} . \supset \therefore y\varepsilon \lim fx \,.=: a\varepsilon\mathrm{q} \,.\, \supset_a . \mathrm{l}_1\, \mathrm{mod}\,\{f[u \frown (a+\mathrm{Q})] - y\} = 0 .$

"Ayant u, f, y la signification connue, la y est une limite de fx, lorsque, quelque soit le nombre a, la limite inférieure des valeurs absolues des différences entre les valeurs de $f[u \frown (a+\mathrm{Q})]$ et y est zéro."

On peut donner une autre forme au résultat de l'élimination de h. Étant u un ensemble de points (nombres), dans plusieurs questions d'analyse on a à considérer la plus petite classe fermée contenant u (selon la nomenclature de M. G. Cantor). On la désigne par Cu, qu'on peut lire "l'ensemble u rendu fermé (*clausus*);" sa définition est (Form. V, §7, P1):

$$(\alpha)\qquad u\varepsilon\mathrm{Kq} \,.\, \supset . Cu = \mathrm{q} \frown \overline{x\varepsilon}\,[\mathrm{l}_1\, \mathrm{mod}\,(u - x) = 0] . \qquad \text{Def.}$$

"On appelle Cu l'ensemble des nombres x tels que la limite inférieure des valeurs absolues des différences entre les nombres u et x soit nulle." L'ensemble Cu contient tous les points de u, et les points limites; il ne faut pas le confondre avec la classe dérivée de u. Commençons par réduire la prop. (α) à la forme plus utile pour notre but ; opérons sur les deux membres de l'égalité par le signe $x\varepsilon$, c'est-à-dire posons ce signe en avant des deux membres. Le premier sera $x\varepsilon Cu$. Le second membre devient $x\varepsilon \{ \mathrm{q} \frown \overline{x\varepsilon}\, [\mathrm{l}_1 \,\mathrm{mod}\, (u - x) = 0] \}$; mais, étant a et b des classes, on a l'identité logique déjà mentionnée $x\varepsilon a \frown b . = . x\varepsilon a . x\varepsilon b$ (Intr. §16, P2) ; donc cette expression se transforme en :

$$x\varepsilon \mathrm{q} . x\varepsilon \overline{x\varepsilon}\, [\mathrm{l}_1 \,\mathrm{mod}\, (u - x) = 0] \,;$$

les deux signes $x\varepsilon$ et $\overline{x\varepsilon}$, qui représentent des opérations inverses, se détruisent (Int. §17, P1, et §28), et il reste

$$x\varepsilon \mathrm{q} . \mathrm{l}_1 \,\mathrm{mod}\, (u - x) = 0 .$$

Donc la définition (α) se transforme en :

(β) $$u\varepsilon \mathrm{Kq} . \supset \therefore x\varepsilon Cu . = . x\varepsilon \mathrm{q} . \mathrm{l}_1 \,\mathrm{mod}\, (u - x) = 0 .$$

Posons ici $f[u \frown (a + \mathrm{Q})]$ au lieu de u, et y à la place de x ; supprimons les propositions $f[u \frown (a + \mathrm{Q})]\, \varepsilon \mathrm{Kq}$, et $y\varepsilon \mathrm{q}$, contenues dans les hypothèses, et l'on a, en envertissant les deux membres :

(γ) $$\mathrm{l}_1 \,\mathrm{mod}\, \{ f[u \frown (a + \mathrm{Q})] - y \} = 0 : = . y\varepsilon Cf[u \frown (a + \mathrm{Q})] .$$

Substituons ; la P9 devient :

10. $$\mathrm{HpP1} . \supset \therefore y\varepsilon \lim fx . = : a\varepsilon \mathrm{q} . \supset_a . y\varepsilon Cf[u \frown (a + \mathrm{Q})] .$$

“La condition nécessaire et suffisante pour que y soit une limite de fx, est que, quelque soit le nombre a, y appartienne toujours à l'ensemble $f[u \frown (a + \mathrm{Q})]$ rendu fermé.”

On a ainsi éliminé la lettre h, et l'on a obtenu les prop. 9 et 10, par une nouvelle convention, de la limite inférieure, ou de la classe fermée. Maintenant il n'y a plus dans le second membre de la 10 que la lettre apparente a ; pour la éliminer il faut une nouvelle convention ; cette nouvelle convention est la définition même de la limite. En effet la proposition 10 dit qu'on écrit le premier membre qui ne contient pas a, au lieu du deuxième où figure cette lettre apparente.

Jusqu'à présent nous avons défini la proposition $y\varepsilon \lim fx$, dans les hypothèses de la P1. On n'a pas défini le signe $\lim fx$. Pour bien voir la classe qu'on a défini, prenons la définition sous la forme de la proposition 10. Étant

a, b, c, d des propositions, l'expression $ab \supset . c = d$ est identique à $a \supset . bc = bd$; on passe de la première à la deuxième en *important* b; et en *exportant* b on a la transformation inverse (Form. I, §1, P45). Dans la P10 importons l'hypothèse $y \varepsilon \mathrm{q}$, qu'y est contenue; on a:

$$u \varepsilon \mathrm{Kq} . \mathrm{l}'u = \infty . f \varepsilon \mathrm{q f} u . \supset \therefore y \varepsilon \mathrm{q} . y \varepsilon \lim fx . = : y \varepsilon \mathrm{q} : a \varepsilon \mathrm{q} . \supset_a . y \varepsilon Cf [u \frown (a + \mathrm{Q})] .$$

Opérons les deux membres de l'égalité par le signe $\overline{y\varepsilon}$. À gauche, par des transformations que nous avons déjà rencontrées, on obtient $\mathrm{q} \frown \lim fx$; à droite $\overline{y\varepsilon}\, y \varepsilon \mathrm{q}$ se réduit à q; et l'on a:

11. $u \varepsilon \mathrm{Kq} . \mathrm{l}'u = \infty . f \varepsilon \mathrm{q f} u . \supset . \mathrm{q} \frown \lim fx = \mathrm{q} \frown \overline{y\varepsilon} \{ a \varepsilon \mathrm{q} . \supset_a . y \varepsilon Cf [u \frown (a + \mathrm{Q})] \} .$

Donc la classe qu'on a défini est $\mathrm{q} \frown \lim fx$, les nombres finis qui sont des limites de fx.

Maintenant nous allons définir les propositions $\infty \varepsilon \lim fx$, et $-\infty \varepsilon \lim fx$:

12. HpP11. $\supset :: \infty \varepsilon \lim fx . = \therefore a, m \varepsilon \mathrm{q} . \supset_{a, m} : x \varepsilon u . x > a . fx > m . \sim =_x \Lambda .$ Def.

"Ayant u et f la signification connùe, nous dirons que l'∞ est une valeur limite de fx, lorsque, quels que soient les nombres a et m, il y a toujours une valeur de x, appartenant à l'ensemble u, plus grande que a, qui rend fx supérieure à m."

Cette proposition se transforme comme la P1. La condition

$x \varepsilon u . x > a . fx > m . \sim =_x \Lambda$ se transforme en $x \varepsilon u \frown (a + \mathrm{Q}) . fx \varepsilon m + \mathrm{Q} . \sim =_x \Lambda$,

et en éliminant la x, par l'identité logique déjà rencontrée (Int. §29, P5), en $f [u \frown (a + \mathrm{Q})] \frown (m + \mathrm{Q}) \sim = \Lambda$. Donc la P12 se transforme dans la P13, analogue à la P5,

13. HpP11. $\supset \therefore \infty \varepsilon \lim fx . = : a, m \varepsilon \mathrm{q} . \supset_{a, m} . f [u \frown (a + \mathrm{Q})] \frown (m + \mathrm{Q}) \sim = \Lambda .$

Le second membre de cette égalité, en *exportant* l'hypothèse $a \varepsilon \mathrm{q}$, devient

$$a \varepsilon \mathrm{q} . \supset_a : m \varepsilon \mathrm{q} . \supset_m . f [u \frown (a + \mathrm{Q})] \frown (m + \mathrm{Q}) \sim = \Lambda .$$

Or, étant u une classe de q, par la définition de la limite supérieure, on a (Form. V, §3, P5),

$$\mathrm{l}'u = \infty . = : m \varepsilon \mathrm{q} . \supset_m . u \frown (m + \mathrm{Q}) \sim = \Lambda ,$$

" la limite supérieure des u est infinie, lorsque, quel que soit le nombre m, il y a toujours des nombres de la classe u plus grands que m." Substituons ici $f\,[u \frown (a+\mathrm{Q})]$ au lieu de u; en envertissant les deux membres, on a :

$$m\varepsilon\mathrm{q}\,.\supset_m.\,f\,[u \frown (a+\mathrm{Q})] \frown (m+\mathrm{Q}) \sim = \Lambda :=.\, \mathrm{l}'f\,[u \frown (a+\mathrm{Q})] = \infty .$$

Substituons dans la P13 ; elle se transforme en :

14. $\mathrm{HpP11}.\supset\therefore \infty\,\varepsilon\,\lim fx\,.=:a\varepsilon\mathrm{q}\,.\supset_a.\,\mathrm{l}'f\,[u \frown (a+\mathrm{Q})] = \infty .$

" L' ∞ est une valeur limite de fx, si, quel que soit le nombre a, la limite supérieure des valeurs de la fonction, lorsque la variable prend les valeurs suivantes a, est infinie." On a ainsi éliminé la m. Cette proposition est l'analogue de la P9.

Nous voulons trouver la signification de la proposition "l' ∞ n'est pas une limite de fx." Le second membre de la P13 est une déduction; transportons la thèse dans le premier membre, en la niant, en vertu de l'identité logique $a\supset b\,.=.\,a \sim b = \Lambda$ (Form. I, §3, P8). La prop. 13 devient:

$$\mathrm{HpP11}.\supset\therefore \infty\,\varepsilon\,\lim fx\,.=:a,\, m\varepsilon\mathrm{q}\,.\,f\,[u \frown (a+\mathrm{Q})] \frown (m+\mathrm{Q}) = \Lambda\cdot =_{a,\,m} \Lambda$$

et, en niant les deux membres de l'égalité (Form. I, §2, P5),

15. $\mathrm{HpP11}.\supset\therefore \infty \sim\varepsilon\,\lim fx\,.=:a,\, m\varepsilon\mathrm{q}\,.\,f\,[u \frown (a+\mathrm{Q})] \frown (m+\mathrm{Q}) = \Lambda\cdot \sim =_{a,\,m} \Lambda\cdot$

"L' ∞ n'est pas une limite de la fonction, lorsqu'on peut déterminer deux nombres a et m, tels que nulle valeur de $f\,[u \frown (a+\mathrm{Q})]$ ne soit plus grande que m."

Analoguement à la P12, on pose par définition :

16. $\mathrm{HpP11}.\supset:: -\infty\,\varepsilon\,\lim fx\,.=\therefore a,\, m\varepsilon\mathrm{q}\,.\supset_{a,\,m}: x\varepsilon u\,.\,x > a\,.\,fx < m\,.\sim =_x \Lambda$

et en éliminant la x, la m, et en niant les deux membres, on a les correspondantes des prop. 13, 14, 15 :

17. $\mathrm{HpP11}.\supset\therefore -\infty\,\varepsilon\,\lim fx\quad .=:a,\, m\varepsilon\mathrm{q}\,.\supset_{a,\,m}.\,f\,[u \frown (a+\mathrm{Q})] \frown (m-\mathrm{Q}) \sim = \Lambda,$
18. " $.\supset\therefore$ " $.=:a\varepsilon\mathrm{q}\,.\supset_a.\,\mathrm{l}_1 f\,[u \frown (a+\mathrm{Q})] = -\infty,$
19. " $.\supset\therefore -\infty \sim\varepsilon\,\lim fx\,.=:a,\, m\varepsilon\mathrm{q}\,.\,f\,[u \frown (a+\mathrm{Q})] \frown (m-\mathrm{Q}) = \Lambda\cdot \sim =_{a,\,m} \Lambda\cdot$

Nous avons donc défini (P1) la proposition $y\varepsilon\,\lim fx$, où y est un nombre fini (voir la P11). Puis nous avons défini (P12) la proposition $\infty\varepsilon\,\lim fx$; c'est-à-dire la proposition $y\varepsilon\,\lim fx$, lorsque $y = \infty$. Examinons bien ce passage ; on a :

(α) $$y = \infty\,.\supset: y\varepsilon\,\lim fx\,.=.\,\infty\,\varepsilon\,\lim fx$$

par l'identité logique $a = b . \supset : a \varepsilon u . = . b \varepsilon u$, conséquence immédiate de la P10 du §4. partie I du Form. Multiplions (logiquement) les membres des déductions P12 et (α); c'est-à-dire appliquons la formule $a \supset b . c \supset d : \supset . ac \supset bd$ (Form. 1, §1, P30); on a :

$$(\beta) \qquad \text{HpP11. } y = \infty . \supset :: y \varepsilon \lim fx . = . \infty \varepsilon \lim fx :: \infty \varepsilon \lim fx . = \therefore \text{ etc.}$$

en indiquant par etc ce qui est à droite du signe $= \therefore$ dans la définition P12; de la Thèse de la (β) on a $y \varepsilon \lim fx . = \therefore$ etc (par l'identité $a = b . b = c . \supset . a = c$); donc, par le *syllogisme*

$$(\gamma) \qquad \text{HpP11. } y = \infty . \supset :: y \varepsilon \lim fx . = \therefore \text{ etc.}$$

Analoguement on a défini (p. 16) la proposition $y \varepsilon \lim fx$, lorsque $y = -\infty$. Donc la proposition $y \varepsilon \lim fx$ est définie lorsque $y \varepsilon \mathrm{q} . \smile . y = \infty . \smile . y = -\infty$. La proposition $y = \infty$ s'énonce au moyen du signe ε et du signe ι (signe qui signifie *égal*, *ἴσος*, voir Intr. §31); et devient $y \varepsilon \iota \infty$; la proposition $y = -\infty$ se transforme en $y \varepsilon \iota - \infty$; et l'ensemble

$$y \varepsilon \mathrm{q} . \smile . y = \infty . \smile . y = -\infty$$

devient $y \varepsilon \mathrm{q} . \smile . y \varepsilon \iota \infty . \smile . y \varepsilon \iota - \infty$. Or (Intr. §16, P3) $x \varepsilon a . \smile . x \varepsilon b$ est identique à $x \varepsilon a \smile b$; donc on a défini $y \varepsilon \lim fx$ lorsque $y \varepsilon \mathrm{q} \smile \iota \infty \smile \iota - \infty$. Maintenant il ne convient pas d'étendre la signification de la proposition $y \varepsilon \lim fx$ à d'autres cas; et nous poserons, comme définition du signe $\lim fx$:

$$20. \qquad \text{HpP11. } \supset . \lim fx = (\mathrm{q} \smile \iota \infty \smile \iota - \infty) \frown \lim fx. \qquad \text{Def.}$$

"Par $\lim fx$ nous entendons les nombres finis, ou égaux à $\pm \infty$, qui, selon les définitions 1, 12, 16, sont des valeurs limites de fx."

En développant les définitions de la limite d'une fonction, nous avons fait usage des limites supérieure et inférieure d'un ensemble, et des ensembles fermés. Ces idées sont plus simples que l'idée de la limite d'une fonction. Les idées des limites supérieure et inférieure d'une classe, et aussi des points limites d'une classe, s'expriment au moyen de la seule idée logique de classe; la limite d'une fonction exige encore l'idée de fonction ou de correspondance. En conséquence les auteurs qui définissent les points limites d'un ensemble en se servant de la limite d'une fonction, expliquent une idée facile au moyen des idées plus compliqués; et s'exposent encore à des difficultés, sur lesquelles nous ne voulons pas nous arrêter.

Sont liés à la limite d'une fonction deux classes, que nous indiquerons par $\mu'(f, u)$ et $\mu_1(f, u)$, et que nous allons définir.

21. HpP11. $\supset . \mu'(f, u)$

$$= \mathrm{q} \frown \overline{m\varepsilon}\{a\varepsilon\mathrm{q} . f[u \frown (a + \mathrm{Q})] \frown (m + \mathrm{Q}) = \Lambda . \sim =_a \Lambda . \quad \text{Def.}$$

"Ayant f et u la même signification, par $\mu'(f, u)$ nous désignons tout nombre m tel qu'il y a un nombre a, de façon que nulle valeur de $f[u \frown (a + \mathrm{Q})]$ ne soit plus grande que m."

21'. HpP11. $\supset . \mu_1(f, u)$

$$= \mathrm{q} \frown \overline{m\varepsilon}\{a\varepsilon\mathrm{q} . f[u \frown (a + \mathrm{Q})] \frown (m - \mathrm{Q}) = \Lambda . \sim =_a \Lambda\} . \quad \text{Def.}$$

Au second membre de l'égalité qui figure dans la déf. 21, la lettre a est apparente puisqu'elle figure aussi comme indice au signe $\sim =$. La lettre m est aussi apparente; car la proposition entre $\{\ldots\}$ est une condition contenant la lettre m, et en posant en avant le signe $\overline{m\varepsilon}$, on a une classe indépendante de m (Intr. §17). Donc ce second membre dépend des lettres f et u; et nous l'avons indiqué par $\mu'(f, u)$. Mais, puisque dans cette Note les lettres f et u ne changeront jamais de signification, pour abréger nous écrirons μ' au lieu de $\mu'(f, u)$; et écrirons μ_1 au lieu de $\mu_1(f, u)$.

Opérons sur les deux membres de l'égalité qui figure dans la P21 par le signe $m\varepsilon$; en appliquant les règles que nous avons déjà vues, et en simplifiant l'expression $m\varepsilon\, \overline{m\varepsilon}$ on a:

22. HpP11. $\supset \therefore m\varepsilon\mu' . = : m\varepsilon\mathrm{q} : a\varepsilon\mathrm{q} . f[u \frown (a + \mathrm{Q})] \frown (m + \mathrm{Q}) = \Lambda . \sim =_a \Lambda,$

"m appartient à la classe μ', lorsqu'il est un nombre fini, et qu'on peut déterminer un nombre a tel qu'il n'y a pas de valeur de fx, pour x contenu dans la classe u et supérieur à a, qui soit plus grande que m."

La proposition conditionnelle:

$$(\alpha) \qquad a, m\varepsilon\mathrm{q} . f[u \frown (a + \mathrm{Q})] \frown (m + \mathrm{Q}) = \Lambda . \sim =_{a, m} \Lambda$$

qui figure au second membre de l'égalité dans la P15, est identique à

$$(\beta) \qquad m\varepsilon\mathrm{q} : a\varepsilon\mathrm{q} . f[u \frown (a + \mathrm{Q})] \frown (m + \mathrm{Q}) = \Lambda . \sim =_a \Lambda : \sim =_m \Lambda .$$

C'est l'intuition qui nous dit que les propositions (α) et (β) sont équivalentes;

mais on peut ériger en règle générale cette transformation de la (α) dans la (β). Si a_x et $b_{x,y}$ sont des propositions contenant les lettres qui figurent comme indices, la proposition $a_x b_{x,y} \sim =_{x,y} \Lambda$ "il y a des valeurs de x et de y qui satisfait aux conditions $a_x b_{x,y}$" est identique à la $a_x . b_{x,y} \sim =_y \Lambda . \sim =_x \Lambda$ (ou, en adoptant les parenthèses, $a_x (b_{x,y} \sim =_y \Lambda) \sim =_x \Lambda$) "il y a des valeurs de x qui satisfont à la condition a_x et tels qu'il y a des valeurs de y qui satisfont à la condition $b_{x,y}$. En examinant l'Introduction au Form. on voit que cette transformation est la P3 du §18, dans laquelle on a pris les négatives des deux membres.

Or, ce qui dans la (β) précède le signe $\sim =_m \Lambda$, par la P22, est identique à $m \varepsilon \mu'$. Donc la (β) se transforme en

$$(\gamma) \qquad m \varepsilon \mu' . \sim =_m \Lambda$$

la quelle est identique à

$$(\delta) \qquad \mu' \sim = \Lambda,$$

par une identité logique contenue dans Form. I, §4, P6 (Intr. §16, P5). Substituons enfin dans la P15 au second membre (α) sa valeur (δ) ainsi transformée; on a

23. $$\mathrm{HpP11} . \supset : \infty \sim \varepsilon \lim fx . = . \mu' \sim = \Lambda .$$

"L'infini n'est pas une limite de fx, lorsque la classe μ' existe, et réciproquement." Analoguement

23'. $$\mathrm{HpP11} . \supset : - \infty \sim \varepsilon \lim fx . = . \mu_1 \sim = \Lambda .$$

Nous allons prouver que

24. $$\mathrm{HpP11} . \; m' \varepsilon \mu' . m_1 \varepsilon \mu_1 . \supset . m' \geq m_1 .$$

"Tout nombre m' de la classe μ' est supérieur, ou égal, á tout nombre m_1 de la classe μ_1." En effet, substituons à $m' \varepsilon \mu' . m_1 \varepsilon \mu_1$ les valeurs qu'on tire des définitions de μ' et μ_1, P22. En sousentendant toujours HpP11, qui explique la signification des lettres u et f, on a:

$$(\alpha) \quad m' \varepsilon \mu' . m_1 \varepsilon \mu_1 . = : m' \varepsilon \mathrm{q} : a \varepsilon \mathrm{q} . f[u \frown (a + \mathrm{Q})] \frown (m' + \mathrm{Q}) \\ = \Lambda . \sim =_a \Lambda : m_1 \varepsilon \mathrm{q} : a \varepsilon \mathrm{q} . f[u \frown (a + \mathrm{Q})] \frown (m_1 - \mathrm{Q}) = \Lambda . \sim =_a \Lambda .$$

Groupons différemment les propositions du second membre de la (α), ce qui est permis par les propriétés commutative et associative de la multiplication logique.

Et ensuite appliquons la règle de logique (Int. §18, P7) qui transforme l'ensemble des propositions

$$a_x \sim =_x \Lambda . b_x \sim =_x \Lambda ,$$

"il y a des x qui satisfont à la condition a_x, et il y a des x qui satisfont à la condition b_x" dans sa équivalente

$$a_y b_z \sim =_{y,\, z} \Lambda ,$$

"il y a une valeur de y et une de z qui satisfont aux conditions $a_y b_z$." (On se tromperait en la transformant en $a_x b_x \sim =_x \Lambda$.) Dans notre cas nous avons deux propositions qui contiennent comme indices la même lettre a; nous en ferons une proposition seule en écrivant deux lettres différentes a' et a_1 dans les deux formules, et l'on a :

$$(\beta) \quad m' \varepsilon \mu' . m_1 \varepsilon \mu_1 . = : m', m_1 \varepsilon \mathrm{q} : a', a_1 \varepsilon \mathrm{q} . f[u \frown (a' + \mathrm{Q})] \frown (m' + \mathrm{Q}) \\ = \Lambda . f[u \frown (a_1 + \mathrm{Q})] \frown (m_1 - \mathrm{Q}) = \Lambda . \sim =_{a',\, a_1} \Lambda ,$$

"dire que m' et m_1 appartiennent respectivement aux classes μ' et μ est identique à dire qu'ils sont des nombres finis, et qu'on peut déterminer deux nombres a' et a_1 tels que nulle valeur de $f[u \frown (a' + \mathrm{Q})]$ ne soit plus grande que m', et nulle valeur de $f[u \frown (a_1 + \mathrm{Q})]$ ne soit plus petite que m_1."

$$(\gamma) \quad m', m_1 \varepsilon \mathrm{q} . a', a_1 \varepsilon \mathrm{q} . f[u \frown (a' + \mathrm{Q})] \frown (m' + \mathrm{Q}) \\ = \Lambda . f[u \frown (a_1 + \mathrm{Q}) \frown (m_1 - \mathrm{Q}) = \Lambda . \supset : x \varepsilon u \frown (a' + \mathrm{Q}) \frown (a_1 + \mathrm{Q}) . \sim =_x \Lambda .$$

"Or, étant m' et m_1 des quantités, et a', a_1 des quantités qui verifient aux conditions dont on a parlé précédemment, on peut déterminer un nombre x appartenant à la classe u, plus grand que a' et que a_1." En effet, puisque l'$u = \infty$, il y a dans la classe u des nombres supérieurs à a' et à a_1.

$$(\delta) \qquad \mathrm{Hp}(\gamma) . \ x \varepsilon u \frown (a' + \mathrm{Q}) \frown (a_1 + \mathrm{Q}) . \supset . fx \leq m' .$$

"En conservant les hypoth. de (γ), et si u est un nombre dont on vient de parler, fx ne supère pas m'." Si l'on désire analyser ce passage, remarquons que $f[u \frown (a' + \mathrm{Q})] \frown (m' + \mathrm{Q}) = \Lambda$, en transportant le second facteur dans le second membre, et en posant $m' - \mathrm{Q}_0$ au lieu de $\sim (m' + \mathrm{Q})$, se transforme en $f[u \frown (a' + \mathrm{Q})] \supset m' - \mathrm{Q}_0$; maintenant des hypothèses

$$x \varepsilon u \frown (a' + \mathrm{Q}) . f[u \frown (a' + \mathrm{Q})] \supset m' - \mathrm{Q}_0 ,$$

on déduit

$$fx \, \varepsilon \, m' - \mathrm{Q}_0 ,$$

comme resulte de l'identité logique $x \varepsilon u . f u \supset v . \supset . f x \varepsilon v$, (conséquence immédiate de Intr. §29, P3) et $fx \varepsilon m' - Q_0$ est identique à la thèse $fx \leq m'$. Analoguement

$$(\varepsilon) \qquad \mathrm{Hp}(\delta). \supset . fx \geq m_1 .$$

Composons les (δ) et (ε), qui ont la même hypothèse:

$$(\zeta) \qquad \mathrm{Hp}(\delta) . \supset . fx \leq m' . fx \geq m_1 .$$

Or de la Thèse de (ζ), par l'arithmétique on a $m' \geq m$;

$$(\eta) \qquad \mathrm{Ths}(\zeta) . \supset . m' \geq m_1 ,$$

et la thèse de (η) est la thèse du théorème 24 à démontrer. Dans les démonstrations qu'on fait avec le langage ordinaire, on s'arrête habituellement à ce point; mais nous voulons, par des transformations successives, obtenir toute la P24. À cet effet, les (ζ) et (η) sont les prémisses d'un *syllogisme* (Form. I, §1, P13), dont la conclusion est $\mathrm{Hp}(\zeta) \supset \mathrm{Ths}(\eta)$, ou, en développant:

$$(\theta) \qquad \mathrm{Hp}(\gamma) . x \varepsilon u \frown (a' + Q) \frown (a_1 - Q) . \supset . m' \geq m_1 .$$

Ici l'Hyp. contient la lettre x, qui ne figure pas dans la Thèse; on la élimine par la loi que une deduction $a_{x,y} \supset_{x,y} b_x$, où l'Hp. contient la lettre y qui ne figure pas dans la Ths., est identique à la $a_{x,y} \sim =_y \Lambda . \supset_x b_x$: "S'il y a des valeurs de y qui satisfont à la condition $a_{x,y}$, alors est vraie la b_x" (Intr. §18, P10). Par cette transformation la (θ) devient

$$(\iota) \qquad \mathrm{Hp}(\gamma) : x \varepsilon u \frown (a' + Q) \frown (a_1 - Q) . \sim =_x \Lambda : \supset . \mathrm{Ths} \mathrm{P} 24,$$

"dans les Hp. de la (γ), s'il y a un nombre x tel que ...., alors est vraie la Ths. du théorème." Or la (γ) dit que la seconde partie de l'Hp. de la (ι) est conséquence de la première partie; donc on peut supprimer cette seconde partie, par l'identité logique $a \supset b . = . a = ab$ (Form. I, §1, P33). En la supprimant, et en développant l'abréviation $\mathrm{Hp}(\gamma)$ on a:

$$(\kappa) \quad m, m' \varepsilon \mathrm{q} . a', a_1 \varepsilon \mathrm{q} . f[u \frown (a' + Q)] \frown (m' + Q) \\ = \Lambda . f[u \frown (a_1 + Q)] \frown (m_1 - Q) = \Lambda . \supset . \mathrm{Ths} \mathrm{P} 24.$$

L'Hp. de cette proposition contient les lettres a' et a_1 qui ne figurent pas dans la Ths.; on les élimine avec le même procès, et l'on a:

$$(\lambda) \quad m, m' \varepsilon \mathrm{q} : a', a_1 \varepsilon \mathrm{q} . f[u \frown (a' + Q)] \frown (m' + Q) \\ = \Lambda . f[u \frown (a_1 + Q) \frown (m_1 - Q) = \Lambda . \sim =_{a', a_1} \Lambda : \supset . \mathrm{Ths} \mathrm{P} 24.$$

Mais par la (β), l'Hp. de la (λ) est identique à $m'\varepsilon\mu' . m_1\varepsilon\mu_1$; donc :

$$m'\varepsilon\mu' . m_1\varepsilon\mu_1 . \supset . m' \geq m_1 . \qquad \text{c. q. f. d.}$$

Théorème.—"Ayant u et f la même signification, si l'∞ et $-\infty$ ne sont pas des valeurs limites de fx, alors la limite inférieure des nombres μ' et la limite supérieure des nombres μ_1 sont des nombres déterminés et finis, et la première n'est pas inférieure à la seconde" :

25. $\quad$ $\mathrm{HpP11} . \ \infty \sim \varepsilon \lim fx . -\infty \sim \varepsilon \lim fx . \supset . \mathrm{l}_1\mu', \mathrm{l}'\mu_1\varepsilon\mathrm{q} . \mathrm{l}_1\mu' \geq \mathrm{l}'\mu_1 .$

En effet la P23 a la forme $a\supset . b = c$, la quelle est identique à l'ensemble $ab\supset c$ et $ac\supset b$; comme nous avons déjà dit pour expliquer la transformation de la P1 dans les P2 et P3. Nous écrirons seulement la première des deux propositions, la quelle est :

(α) $\quad$ $\mathrm{HpP11} . \ \infty \sim \varepsilon \lim fx . \supset . \mu' \sim = \Lambda .$

De la P23′ on tire analoguement

(α') $\quad$ $\mathrm{HpP11} . \ -\infty \sim \varepsilon \lim fx . \supset . \mu_1 \sim = \Lambda .$

La P24, en *exportant* HpP11, devient :

(β) $\quad$ $\mathrm{HpP11} . \ \supset : m'\varepsilon\mu' . m_1\varepsilon\mu_1 . \supset_{m', m_1} . m' \geq m_1 .$

Multiplions logiquement membre à membre les propositions (α) (α') et (β) (Form. I, §1, P30) ; en *simplifiant*, c'est-à-dire en écrivant une seule fois les facteurs logiques répétés, comme dit l'identité logique $aa = a$ (Form. I, §1, P6), on a :

(γ) $\quad$ $\mathrm{HpP11} . \ \infty \sim \varepsilon \lim fx . -\infty \sim \varepsilon \lim fx . \supset :$

$$\mu' \sim = \Lambda . \mu_1 \sim = \Lambda : m'\varepsilon\mu' . m_1\varepsilon\mu_1 . \supset_{m', m_1} . m' \geq m_1 .$$

Or dans la théorie des limites supérieures et inférieures des ensembles, on a la proposition :

(δ) $\quad$ $u, v\varepsilon\mathrm{Kq} . u \sim = \Lambda . v \sim = \Lambda : m'\varepsilon u . m_1\varepsilon v . \supset_{m', m_1} m' \geq m_1 : \supset . \mathrm{l}_1 u, \mathrm{l}'v\varepsilon\mathrm{q} . \mathrm{l}_1 u \geq \mathrm{l}'v .$

"Étant u et v des ensembles de nombres, effectivement existantes, si chaque nombre de u est supérieur, ou égal, à chaque nombre de v, alors la limite inférieure des u et la limite supérieure des v ont des valeurs finies, et la première est supérieure, ou égale, à la deuxième." Cette proposition est une conséquence de Form. V, §3, P4, P4′ ; mais elle n'est pas explicitement contenue dans le formulaire publié, et il sera bien de l'ajouter. Si, dans la (δ) au lieu de u et v on lit μ' et μ_1, elle devient

(ϵ) $$\mathrm{Ths}(\gamma) \supset \mathrm{Ths}\mathrm{P}25.$$

Mais on peut mettre la (γ) sous la forme

(ζ) $$\mathrm{Hp}\mathrm{P}25 \supset \mathrm{Ths}(\gamma).$$

Les (ϵ) et (ζ) sont les prémisses d'un syllogisme, dont la conséquence est

$$\mathrm{Hp}\mathrm{P}25 \supset \mathrm{Ths}\mathrm{P}25.$$

Théorème.—"Si la limite inférieure des nombres μ' est finie (comme cela arrive dans les hypothèses de la proposition précédente), elle est une valeur limite de la fonction fx":

26. $$\mathrm{Hp}\mathrm{P}11.\ \mathrm{l}_1\mu'\varepsilon \mathrm{q} \,.\supset.\, \mathrm{l}_1\mu'\varepsilon \lim fx.$$

En effet, appellons l cette limite inférieure, c'est-à-dire posons

(α) $$l = \mathrm{l}_1\mu'.$$

La définition de la limite inférieure d'une classe est (Form. V, §3, P1'),

(β) $$u\varepsilon\mathrm{Kq}\,.\,l\varepsilon\mathrm{q}\,.\supset::l=\mathrm{l}_1u\,.=\therefore u\frown(l-\mathrm{Q})=\Lambda:y\varepsilon l+\mathrm{Q}\,.\supset_y.\,u\frown(y-\mathrm{Q})\sim=\Lambda.$$

"Étant u un ensemble de nombres, et l un nombre fini, on dit que l est la limite inférieure des u, lorsqu'il n'y a pas de nombre u inférieur à l; mais, quel que soit le nombre y supérieur à l, il y a toujours des nombres u inférieurs à y." Posons dans la (β) μ' au lieu de u; par HpP26, hypothèses que dans cette démonstration nous sousentendons toujours, l'hyp. de (β) est vraie, et on ne l'écrit plus; et la (α) se transforme dans l'ensemble des deux propositions (γ) et (δ):

(γ) $$\mu'\frown(l-\mathrm{Q})=\Lambda,$$

(δ) $$y\varepsilon l+\mathrm{Q}\,.\supset_y.\,\mu'\frown(y-\mathrm{Q})\sim=\Lambda.$$

Transformons d'abord la (γ). Opérons par le signe $m\varepsilon$; on a:

(ϵ) $$m\varepsilon\mu'.\,m\varepsilon l-\mathrm{Q}\,.=_m\Lambda.$$

Substituons à $m\varepsilon\mu'$ sa valeur donnée par la P22; remarquons tout de suite que $m\varepsilon\mathrm{q}\,.\,m\varepsilon l-\mathrm{Q}$ se réduisent à $m\varepsilon l-\mathrm{Q}$, car la première condition est contenue dans la seconde; on a:

(ζ) $$m\varepsilon l-\mathrm{Q}:a\varepsilon\mathrm{q}\,.\,f[u\frown(a+\mathrm{Q})]\frown(m+\mathrm{Q})=\Lambda\,.\sim=_a\Lambda:=_m\Lambda.$$

Or la proposition $a_x . b_{xy} \sim =_y \Lambda . =_x \Lambda$, "il n'y a pas de valeur de x qui satisfasse à la condition a_x, et telle qu'on puisse déterminer un y qui satisfasse à la condition $b_{x,y}$" est identique à la proposition $a_x b_{xy} =_{x,y} \Lambda$, "il n'y a pas de valeurs de x et de y qui satisfassent aux conditions $a_x b_{xy}$" (Intr. §18, P3). Appliquons cette transformation à la (ζ), en supposant que x soit m, y la a, que a_x représente la $m \varepsilon l - Q$, et b_{xy} toute la proposition entre (:). On a:

$$(\eta) \qquad m\varepsilon l - Q . a\varepsilon q . f[u \frown (a + Q)] \frown (m + Q) = \Lambda . =_{a,m} \Lambda .$$

Transportons le troisième facteur dans le second membre, par l'identité logique $ab = \Lambda . = . a \supset \sim b$; on a:

$$(\theta) \qquad m\varepsilon l - Q . a\varepsilon q . \supset_{a,m} . f[u \frown (a + Q)] \frown (m + Q) \sim = \Lambda .$$

Posons $l - h$ au lieu de m; la $m\varepsilon l - Q$ devient $l - h\varepsilon l - Q$, ou $h\varepsilon Q$, et on a:

$$(\iota) \qquad h\varepsilon Q . a\varepsilon q . \supset_{a,h} . f[u \frown (a + Q)] \frown (l - h + Q) \sim = \Lambda .$$

"Quel que soit le nombre positif h, et quel que soit le nombre a, il y a toujours de valeurs de $f[u \frown (a + Q)]$ plus grandes que $l - h$."

Maintenant transformons la prop. (δ). Posons $l + h$ au lieu de y; on a:

$$(\varkappa) \qquad h\varepsilon Q . \supset_h . \mu' \frown (l + h - Q) \sim = \Lambda .$$

Opérons la thèse par $m\varepsilon$; substituons à $m\varepsilon\mu'$ sa valeur donnée par P22; réduisons les deux propositions $m\varepsilon q . m\varepsilon l + h - Q$ à la dernière [cfr. la transformation de la (γ) en (ζ)]; la ($\varkappa$) devient

$$(\lambda) \quad h\varepsilon Q . \supset_h \therefore m\varepsilon l + h - Q : a\varepsilon q . f[u \frown (a + Q)] \frown (m + Q) = \Lambda . \sim =_a \Lambda : \sim =_m \Lambda .$$

Or une proposition de la forme $a_x : b_y . c_{x,y} . \sim =_y \Lambda : \sim =_x \Lambda$ (il y a des x qui satisfont à la condition a_x et tels qu'il y a des y qui satisfont aux conditions $b_y . c_{x,y}$) est identique à la proposition $b_y : a_x . c_{x,y} . \sim =_x \Lambda : \sim =_y \Lambda$, transformation analogue à celles que nous avons déjà appliquées plusieurs fois (Intr. §18). Appliquons cette transformation à la thèse de la (γ); les lettres x et y sont ici m et a; a_x est $m\varepsilon l + h - Q$, b_y est $a\varepsilon q$, et $c_{x,y}$ est $f[u \frown (a + Q)] \frown (m + Q) = \Lambda$; la ($\lambda$) se transforme en:

$$(\mu) \quad h\varepsilon Q . \supset_h \therefore a\varepsilon q : m\varepsilon l + h - Q . f[u \frown (a + Q)] \frown (m + Q) = \Lambda . \sim =_m \Lambda : \sim =_a \Lambda .$$

Or on a:

$$(\nu) \quad a\varepsilon q . m\varepsilon l + h - Q . f[u \frown (a + Q)] \frown (m + Q) = \Lambda . \supset . f[u \frown (a + Q)] \supset l + h - Q .$$

"Étant a un nombre, m un nombre plus petit que $l+h$, s'il n'y a pas de valeur de $f[u \frown (a+\mathrm{Q})]$ supérieure à m, alors toutes les valeurs de $f[u \frown (a+\mathrm{Q})]$ sont inférieures à $l+h$." En effet $f[u \frown (a+\mathrm{Q})] \frown (m+\mathrm{Q}) = \Lambda$ est identique à $f[u \frown (a+\mathrm{Q})] \supset m - \mathrm{Q}_0$; la classe $m - \mathrm{Q}_0 \supset l+h-\mathrm{Q}$, tout nombre non supérieur à m est inférieur à $l+h$; donc $f[u \frown (a+\mathrm{Q})] \supset l+h-\mathrm{Q}$. Dans la (ν), l'Hyp. contient la lettre m, qui ne figure pas dans la Thèse; on la élimine, selon les règles connues (Int. §18, P10), et l'on a:

$$(\xi) \quad a \varepsilon \mathrm{q} : m \varepsilon l + h - \mathrm{Q} \,.\, f[u \frown (a+\mathrm{Q})] \frown (m+\mathrm{Q}) = \Lambda \,.\, \sim =_m \Lambda : \supset . \\ f[u \frown (a+\mathrm{Q})] \supset l+h-\mathrm{Q}.$$

Or étant a et b des propositions, de $a \supset b$ on déduit $a \sim = \Lambda \,.\, \supset .\, b \sim = \Lambda$ (Form. I, §3, P14). Donc, en opérant sur les deux membres de la (ξ) par le signe $\sim =_a \Lambda$, on a:

$$(o) \qquad \mathrm{Ths}(\mu) \,.\, \supset : a \varepsilon \mathrm{q} \,.\, f[u \frown (a+\mathrm{Q})] \supset l+h-\mathrm{Q} \,.\, \sim =_a \Lambda.$$

Par le syllogisme, de (μ) et (o) on tire

$$(\pi) \qquad h \varepsilon \mathrm{Q} \,.\, \supset_h : a \varepsilon \mathrm{q} \,.\, f[u \frown (a+\mathrm{Q})] \supset l+h-\mathrm{Q} \,.\, \sim =_a \Lambda.$$

"Quel que soit le nombre positif h, on peut déterminer un nombre a tel que toutes les valeurs de la fonction, pour $x > a$, soient plus petites que $l+h$."

Nous avons donc décomposé la (α) dans l'ensemble $(\gamma)(\delta)$, transformé la (γ) en (ι) et la (δ) en (π); nous allons recomposer les (ι) et (π). Soit h une quantité positive, a une quantité choisie de façon que toutes les valeurs de $f[u \frown (a+\mathrm{Q})]$ soient plus petites que $l+h$; soit b une quantité quelconque, appellons c le plus grand des deux nombres a et b. Sera c un nombre; donc, en substituant c à a dans (ι) on a:

$$(\rho) \quad h \varepsilon \mathrm{Q} \,.\, a \varepsilon \mathrm{q} \,.\, f[u \frown (a+\mathrm{Q})] \supset l+h-\mathrm{Q} \,.\, b \varepsilon \mathrm{q} \,.\, c = \max(a, b) \,.\, \supset . \\ f[u \frown (c+\mathrm{Q}) \frown (l-h+\mathrm{Q}) \sim = \Lambda,$$

"il y a des valeurs de $f[u \frown (c+\mathrm{Q})]$ plus grandes que $l-h$." Or, puisque $c \geq a$, on a $c + \mathrm{Q} \supset a + \mathrm{Q}$, tout nombre supérieur à c est aussi supérieur à a; en multipliant par u:

$$u \frown (c+\mathrm{Q}) \supset u \frown (a+\mathrm{Q});$$

opérons par le signe f (Form. I, §5, P5); la relation subsiste dans le même sens:

$$f[u \frown (c+\mathrm{Q})] \supset f[u \frown (a+\mathrm{Q})].$$

Mais par $\mathrm{Hp}(\rho)$, $f[u \frown (a+\mathrm{Q})] \supset l+h-\mathrm{Q}$; donc

$$(\sigma) \qquad \mathrm{Hp}(\rho) . \supset . f[u \frown (c+\mathrm{Q})] \supset l+h-\mathrm{Q},$$

"toutes les valeurs de $f[u \frown (c+\mathrm{Q})]$ sont plus petites que $l+h$." On peut aussi écrire cette déduction sous la forme

$$f[u \frown (c+\mathrm{Q})] = f[u \frown (c+\mathrm{Q})] \frown (l+h-\mathrm{Q}),$$

par l'identité logique $a \supset b . = . a = ab$ (Form. I, §1, P33). Substituons dans le second membre de la (ρ) à $f[u \frown (c+\mathrm{Q})]$ sa valeur qu'on vient de trouver ; on a

$$(\tau) \qquad \mathrm{Hp}(\rho) . \supset . f[u \frown (c+\mathrm{Q})] \frown (l+h-\mathrm{Q}) \frown (l-h+\mathrm{Q}) \sim = \Lambda .$$

On peut remplacer $(l+h-\mathrm{Q}) \frown (l-h+\mathrm{Q})$ par l'intervalle $(l-h)^{-}(l+h)$; et l'on a :

$$(\upsilon) \qquad \mathrm{Hp}(\rho) . \supset . f[u \frown (c+\mathrm{Q})] \frown (l-h)^{-}(l+h) \sim = \Lambda,$$

"il y a des valeurs de $f[u \frown (c+\mathrm{Q})]$ appartenant à l'intervalle de $l-h$ à $l+h$;" d'autre coté on a

$$c \geq b,$$

d'où

$$c+\mathrm{Q} \supset b+\mathrm{Q},$$
$$u \frown (c+\mathrm{Q}) \supset u \frown (b+\mathrm{Q}),$$
$$f[u \frown (c+\mathrm{Q})] \supset f[u \frown (b+\mathrm{Q})],$$
$$f[u \frown (c+\mathrm{Q})] \frown (l-h)^{-}(l+h) \supset f[u \frown (b+\mathrm{Q})] \frown (l-h)^{-}(l+h).$$

Faisons suivre les deux membres par le signe $\sim = \Lambda$; la relation subsiste dans le même sens (Form. I, §3, P14) ; on obtient

$$(\phi) \qquad \mathrm{Ths}(\upsilon) . \supset . f[u \frown (b+\mathrm{Q})] \frown (l-h)^{-}(l+h) \sim = \Lambda .$$

Des premisses (υ) et (ϕ), par le *syllogisme*, on a $\mathrm{Hp}(\rho) \supset \mathrm{Ths}(\phi)$, ou, en développant

$$(\chi) \quad h \varepsilon \mathrm{Q} . a \varepsilon \mathrm{q} . f[u \frown (a+\mathrm{Q})] \supset l+h-\mathrm{Q} . b \varepsilon \mathrm{Q} . c = \max(a, b) . \supset . \\ f[u \frown (b+\mathrm{Q})] \frown (l-h)^{-}(l+h) \sim = \Lambda .$$

L'Hp(χ) contient les lettres a et c qui ne figurent pas dans la thèse. On élimine c en supprimant la proposition $c = \max(a, b)$, car étant a et b deux nombres, il y a toujours le plus grand. On élimine a par la règle connue (Int. §18, P10), et l'on a :

$$(\psi) . \quad h \varepsilon \mathrm{Q} : a \varepsilon \mathrm{q} . f[u \frown (a+\mathrm{Q})] \supset l+h-\mathrm{Q} . \sim =_a \Lambda : b \varepsilon \mathrm{Q} . \supset . \mathrm{Ths}(\chi).$$

Mais la deuxième partie de cette hypothèse est contenue dans la première, comme dit la (π); on peut donc la supprimer, par la règle $a \supset b \,.=.\, a = ab$; on a:

(ω) $$h \varepsilon \mathrm{Q} \,.\, b \varepsilon \mathrm{Q} \,.\supset.\, f[u \frown (b + \mathrm{Q})] \frown (l-h) {}^{-} (l+h) \sim \,=\Lambda.$$

Mais, par la P6, la (ω) est identique à.

$$l \varepsilon \lim fx,$$

qui est la proposition à démontrer.

Analoguement on a:

26'. $$\mathrm{HpP11.}\ \mathrm{l}'\mu_1 \varepsilon \mathrm{q} \,.\supset.\, \mathrm{l}'\mu_1 \varepsilon \lim fx.$$

Il suffit en effet de faire quelques changements des signes + en — et l' en l_1 dans la démonstration de la P26 pour démontrer la 26'. Mais, puisque cette démonstration est assez longue, et dans cette Note nous nous proposons d'étudier les différentes formes de raisonnement, on peut déduire la 26' de la 26 par la transformation suivante. On a (Form. V, §3, P16),

$$\mathrm{l}'\mu_1 = -\mathrm{l}_1(-\mu_1),$$

ou, en écrivant toutes les lettres, qu'on ne peut pas ici sousentendre (voir P21 et P21'),

(α) $$\mathrm{l}'\mu_1(f, u) = -\mathrm{l}_1[-\mu_1(f, u)].$$

Mais on reconnait facilement que

(β) $$-\mu_1(f, u) = \mu'(-f, u),$$

"la classe des nombres μ_1, correspondants à la fonction f, et à la classe u, changés de signe, est identique à la classe des nombres μ', correspondants à la fonction $-f$, et à la même classe u." Donc:

(γ) $$\mathrm{l}'\mu_1(f, u) = -\mathrm{l}_1\mu'(-f, u).$$

Maintenant, par la P26, que nous venons de démontrer:

(δ) $$\mathrm{l}_1\mu'(-f, u) \,\varepsilon \lim_{x, u, \infty} -fx.$$

Il est aussi facile de prouver que

(ε) $$\lim_{x, u, \infty} -fx = -\lim_{x, u, \infty} fx.$$

De la (γ), (δ) et (ε) on déduit la P26'.

Théorème.—“Ayant u et f la même signification, si ni ∞ ni $-\infty$ ne sont des limites de fx, alors il y a des valeurs finies, limites de fx.”

27. $$\mathrm{HpP11}.\ \infty \sim\varepsilon \lim fx \,.\, -\infty \sim\varepsilon \lim fx \,.\supset.\, \mathrm{q} \frown \lim fx \sim= \Lambda.$$

En effet, la P25 et la P26 sont :

(α) $$\mathrm{HpP11}.\ \infty \sim\varepsilon \lim fx \,.\, -\infty \sim\varepsilon \lim fx \,.\supset.\, \mathrm{l}_1\mu'\varepsilon\mathrm{q},$$

(β) $$\mathrm{HpP11}.\ \mathrm{l}_1\mu'\varepsilon\mathrm{q} \,.\supset.\, \mathrm{l}_1\mu'\varepsilon \lim fx.$$

On en déduit

(γ) $$\mathrm{HpP11}.\ \infty \sim\varepsilon \lim fx \,.\, -\infty \sim\varepsilon \lim fx \,.\supset.\, \mathrm{l}_1\mu'\varepsilon\mathrm{q} \frown \lim fx.$$

La forme de cette déduction est, que si $ab\supset c \,.\, ac\supset d \,.\supset.\, ab\supset cd$, où $a = \mathrm{HpP11}$, $b = (\infty \sim\varepsilon \lim fx)(-\infty \sim\varepsilon \lim fx)$, $c = (\mathrm{l}_1\mu'\varepsilon\mathrm{q})$, et $d = (\mathrm{l}_1\mu'\varepsilon \lim fx)$. Cette formule de logique n'est pas explicitement contenue dans le Form. ; mais elle est reductible aux formes que y sont contenues. Or, étant u une classe, de $x\varepsilon u$ on déduit $u \sim= \Lambda$, (si x est un individu de la classe u, alors la classe u n'est pas nulle) ; donc :

(δ) $$\mathrm{Ths}(\gamma) \,.\supset.\, \mathrm{q} \frown \lim fx \sim= \Lambda.$$

Les (γ) et (δ) sont les prémisses d'un syllogisme, qui a pour conclusion la proposition à démontrer.

Dans la P27 transportons les Hp $\infty \sim\varepsilon \lim fx \,.\, -\infty \sim\varepsilon \lim fx$ dans le second membre, selon l'identité logique $abc\supset d \,.=.\, a\supset d \smile \sim b \smile \sim c$ (Form. I, §2, P25). On a :

(α) $$\mathrm{HpP11}.\ \supset.\, \mathrm{q} \frown \lim fx \sim= \Lambda \,.\smile.\, \infty \,\varepsilon \lim fx \,.\smile.\, -\infty \,\varepsilon \lim fx.$$

“Ou il y a des nombres finis qui sont des limites de fx, ou l'∞, ou $-\infty$ en sont des valeurs limites.” La prop. $\infty \,\varepsilon \lim fx$ est reductible à la forme $\iota\infty \frown \lim fx \sim= \Lambda$: et la (α) devient :

(β) $$\mathrm{HpP11}.\ \supset.\, \mathrm{q} \frown \lim fx \sim= \Lambda \,.\smile.\, \iota\infty \frown \lim fx \sim= \Lambda \,.\smile.\, (\iota -\infty) \frown \lim fx \sim= \Lambda.$$

La formule $a \sim= \Lambda \,.\smile.\, b \sim= \Lambda \,.=.\, a \smile b \sim= \Lambda$ (Form. I, §3, P9) transforme la (β) en :

(γ) $$\mathrm{HpP11}.\ \supset.\, (\mathrm{q} \frown \lim fx) \smile (\iota\infty \frown \lim fx) \smile (\iota -\infty \frown \lim fx) \sim= \Lambda$$

et la $(a \frown c) \smile (b \frown c) = (a \smile b) \frown c$ (propriété distributive de la multiplication, Form. I, §2, P22), transforme la (γ) en :

(δ) $$\mathrm{HpP11}.\ \supset.\, (\mathrm{q} \smile \iota\infty \smile \iota -\infty) \frown \lim fx \sim= \Lambda.$$

Celle ci, par la définition P10, devient

28. $\mathrm{HpP11}.\supset.\lim fx \sim\, = \Lambda.$

"Toute fonction fx, donnée pour les valeurs de la variable appartenant à un ensemble illimité supérieurement, lorsque la variable tend à l'∞, a des valeurs limites."

Théorème.—"Si un nombre z est supérieur à quelque nombre m de la classe μ', il n'est pas une limite de fx."

29. $\mathrm{HpP11}.\ m\varepsilon\mu'.\, z\varepsilon m + \mathrm{Q}\,.\supset.\, z \sim \varepsilon \lim fx.$

En effet, posons $h = z - m$; h sera une quantité positive. Toute l'intervalle de $z - h$ à $z + h$ sera contenue dans l'intervalle de m à ∞, car

$(z-h)\,{}^{-}\,(z+h) = (z - h + \mathrm{Q}) \frown (z + h - \mathrm{Q})$ par la définition de l'intervalle (Form V, §4, P41),

$(z - h + \mathrm{Q}) \frown (z + h - \mathrm{Q}) \supset z - h + \mathrm{Q}$, par la formule $ab \supset a$ (Form. I, §1, P5),
$z - h + \mathrm{Q} = m + \mathrm{Q}$, puisque $z - h = m$.

Donc:

(α) $m\varepsilon\mathrm{q}\,.\, z\varepsilon m + \mathrm{Q}\,.\, h = z - m\,.\supset.\, h\varepsilon\mathrm{Q}\,.\,(z-h)\,{}^{-}\,(z+h) \supset m + \mathrm{Q}\,.$

Maintenant soit a un nombre tel que $f[u \frown (a+\mathrm{Q})] \frown (m + \mathrm{Q}) = \Lambda$; sera aussi

$$f[u \frown (a + \mathrm{Q})] \frown (z - h)\,{}^{-}\,(z + h) = \Lambda$$

par la formule de logique $a \supset b\,.\, bc = \Lambda\,.\supset.\, ac = \Lambda$ (Form. I, §3, P31); et l'on a:

(β) $\mathrm{HpP11}.\ \mathrm{Hp}(\alpha)\,.\, a\varepsilon\mathrm{q}\,.\, f[u \frown (a + \mathrm{Q})] \frown (m + \mathrm{Q}) = \Lambda\,.\supset.$
$h\varepsilon\mathrm{Q}\,.\, a\varepsilon\mathrm{q}\,.\, f[u \frown (a + \mathrm{Q})] \frown (z - h)\,{}^{-}\,(z + h) = \Lambda.$

Nous avons dans la Ths. répété l'Hp. $a\varepsilon\mathrm{q}$, par la formule de logique $a \supset b\,.\supset.\, a \supset ab$ (Form. I, §1, P29). Or, par la P8 on a:

(γ) $\mathrm{Ths}(\beta)\,.\supset.\, z \sim \varepsilon \lim fx,$

des premisses (β) et (γ) on a, $\mathrm{Hp}(\beta) \supset \mathrm{Ths}(\gamma)$, ou, en développant:

(δ) $\mathrm{HpP11}.\ m\varepsilon\mathrm{q}\,.\, z\varepsilon m + \mathrm{Q}\,.\, h = z - m\,.\, a\varepsilon\mathrm{q}\,.\, f[u \frown (a + \mathrm{Q})] \frown (m + \mathrm{Q}) = \Lambda\,.\supset.$
$z \sim \varepsilon \lim fx.$

On élimine la lettre h, qui ne figure pas dans la Ths., en supprimant la proposition $h = z - m$, qui est sa définition. On élimine a par le procès bien connu (Int. §18, P10),

$$(\varepsilon)\quad \mathrm{HpP11}.\ m \varepsilon \mathrm{q} \,.\, z \varepsilon m + \mathrm{Q} : a \varepsilon \mathrm{q} \,.\, f[u \frown (a + \mathrm{Q})] \frown (m + \mathrm{Q}) = \Lambda \,.\, \sim =_a \Lambda : \supset . \; z \sim \varepsilon \lim f x .$$

Or, par la P22, $m \varepsilon \mathrm{q} : a \varepsilon \mathrm{q} \,.\, f[u \frown (a + \mathrm{Q})] \frown (m + \mathrm{Q}) = \Lambda \,.\, \sim =_a \Lambda$ est identique à $m \varepsilon \mu'$; en substituant, la (ε) se transforme dans la proposition à démontrer.

Dans l'Hp. de la P29 il y a la lettre m qui ne figure pas dans la Ths.; éliminons-la; on obtient:

$$(\alpha)\qquad \mathrm{HpP11}.\ z \varepsilon \mathrm{q} : m \varepsilon \mu' \,.\, m \varepsilon z - \mathrm{Q} \,.\, \sim =_m \Lambda : \supset . \, z \sim \varepsilon \lim f x ,$$

$$(\beta)\qquad \text{“} \qquad . \, \mu' \frown (z - \mathrm{Q}) \sim = \Lambda \,.\, \supset . \qquad \text{“} \qquad \text{“}$$

Or $\mu' \frown (z - \mathrm{Q}) \sim = \Lambda$ “il y a des nombres de l'ensemble μ', plus petits que z” est identique à $z > \mathrm{l}_1 \mu'$, “la z est plus grande que la limite inférieure des μ'.” Substituons dans la (β); on a le théorème

$$30. \qquad \mathrm{HpP11}.\ z \varepsilon \mathrm{q} \,.\, z > \mathrm{l}_1 \mu' \,.\, \supset . \, z \sim \varepsilon \lim f x .$$

“Si z est un nombre plus grand que la limite inférieure des nombres μ', il n'est pas une limite de fx.” Analoguement on a:

$$30'. \qquad \mathrm{HpP11}.\ z \varepsilon \mathrm{q} \,.\, z < \mathrm{l}' \mu_1 \,.\, \supset . \, z \sim \varepsilon \lim f x .$$

On déduit que la $\mathrm{l}_1 \mu'$ et $\mathrm{l}' \mu_1$, lorsqu'elles existent, sont respectivement le maximum et le minimum de la classe $\lim f x$.

Il est intéressant le cas où il y a une seule valeur limite de la fonction. Appellons-la y. Pour indiquer que y est la seule valeur limite de fx, sans introduire des notations nouvelles, il suffit de dire que la classe $\lim f x$ est identique à la classe formée de l'individu y; or cette classe est indiquée par ιy; donc nous écrirons

$$\lim f x = \iota y .$$

Dans la pratique on pourra sousentendre le signe ι; et écrire (comme avons vu) $y \varepsilon \lim f x$ pour indiquer “y est une limite de fx,” et (selon l'habitude) $y = \lim f x$ pour indiquer “y est la limite de fx.” Mais ici nous adopterons la notation complète (voir Int. §31).

Théorème.—"Ayant u et f la signification connue, et étant y un nombre fini, la condition nécessaire et suffisante pour que y soit la limite de fx, est que la limite inférieure des nombres μ', et la limite supérieure des nombres μ_1 coincident avec y":

31. $\quad \mathrm{HpP11}.\ y \varepsilon \mathrm{q} . \supset : \iota y = \lim fx . = . \mathrm{l}_1 \mu' = \mathrm{l}' \mu_1 = y.$

En effet, des HpP31, si $\lim fx = \iota y$, puisque $y \sim = \infty$, c'est-à-dire puisque $\infty \sim \varepsilon \iota y$, en substituant à ιy sa valeur égale $\lim fx$, on a $\infty \sim \varepsilon \lim fx$:

(α) $\quad \mathrm{HpP31}.\ \iota y = \lim fx . \supset . \infty \sim \varepsilon \lim fx.$

Analoguement

(β) $\quad \mathrm{HpP31}.\ \iota y = \lim fx . \supset . -\infty \sim \varepsilon \lim fx,$

Composons les (α) et (β):

(γ) $\quad \mathrm{Hp}(\alpha) . \supset . \infty \sim \varepsilon \lim fx . -\infty \sim \varepsilon \lim fx$

de la P25 on a:

(δ) $\quad \mathrm{Hp}(\alpha) . \mathrm{Ths}(\gamma) . \supset . \mathrm{l}_1 \mu', \mathrm{l}' \mu_1 \varepsilon \mathrm{q}.$

Supprimons ici $\mathrm{Ths}(\gamma)$, qui, par la (γ) est conséquence de $\mathrm{Hp}(\alpha)$; on a:

(ε) $\quad \mathrm{Hp}(\alpha) . \supset . \mathrm{l}_1 \mu', \mathrm{l}' \mu_1 \varepsilon \mathrm{q}.$

Or, en multipliant les propositions 26 et 26′ on a:

(ζ) $\quad \mathrm{Hp}(\alpha) . \mathrm{Ths}(\varepsilon) . \supset . \mathrm{l}_1 \mu', \mathrm{l}' \mu_1 \varepsilon \lim fx.$

Supprimons $\mathrm{Ths}(\varepsilon)$, qui est conséquence de $\mathrm{Hp}(\alpha)$, comme dit la prop. (ε); développons le second membre; on a:

(η) $\quad \mathrm{Hp}(\alpha) . \supset . \mathrm{l}_1 \mu' \varepsilon \lim fx . \mathrm{l}' \mu_1 \varepsilon \lim fx.$

Substituons à $\lim fx$ sa valeur égale ιy, comme dit $\mathrm{Hp}(\alpha)$:

(θ) $\quad \mathrm{Hp}(\alpha) . \supset . \mathrm{l}_1 \mu' \varepsilon \iota y . \mathrm{l}' \mu_1 \varepsilon \iota y.$

Or le groupement des signes $\varepsilon \iota$ est identique au signe $=$; en substituant, et en développant $\mathrm{Hp}(\alpha)$, on a:

(ι) $\quad \mathrm{HpP31}.\ \lim fx = \iota y . \supset . \mathrm{l}_1 \mu' = y . \mathrm{l}' \mu_1 = y.$

Réciproquement, de la P26 on a:

$(\varkappa)$ $\quad \mathrm{HpP31}.\ y = \mathrm{l}_1 \mu' = \mathrm{l}' \mu_1 . \supset . y \varepsilon \lim fx,$

(λ) $\quad \mathrm{Hp}(\varkappa) . \supset . \mathrm{l}_1 \mu', \mathrm{l}' \mu_1 \varepsilon \mathrm{q}.$

Or si une classe a une limite inférieure, ou une limite supérieure elle existe, effectivement :

$$u \varepsilon \mathrm{Kq} \,.\, \mathrm{l}'u \varepsilon \mathrm{q} \,.\supset.\, u \sim = \Lambda.$$

Cette formule n'est pas écrite dans le formulaire ; on fera bien de l'ajouter. Dans notre cas :

(μ) $$\mathrm{Hp}(x) \,.\, \mathrm{Ths}(\lambda) \,.\supset.\, \mu' \sim = \Lambda \,.\, \mu_1 \sim = \Lambda.$$

Supprimons Ths(λ) qu'est conséquence de Hp(x), par la prop. (λ) ; on a :

(ν) $$\mathrm{Hp}(x) \,.\supset.\, \mu' \sim = \Lambda \,.\, \mu_1 \sim = \Lambda.$$

Or (P23 et P23′), $\mu' \sim = \Lambda$ signifie $\infty \sim \varepsilon \lim fx$; et $\mu_1 \sim = \Lambda$ signifie $-\infty \sim \varepsilon \lim fx$; donc :

(ξ) $$\mathrm{Hp}(x) \,.\supset.\, \infty \sim \varepsilon \lim fx \,.\, -\infty \sim \varepsilon \lim fx.$$

La P30 dit dans notre cas :

(o) $$\mathrm{Hp}(x) \,.\, z \varepsilon \mathrm{q} \,.\, z > y \,.\supset.\, z \sim \varepsilon \lim fx.$$

À $z \varepsilon \mathrm{q} \,.\, z > y$ substituons $z \varepsilon y + \mathrm{Q}$; transportons le deuxième membre, et exportons Hp(x) ; la (o) devient

(π) $$\mathrm{Hp}(x) \,.\supset:\, z \varepsilon y + \mathrm{Q} \,.\, z \varepsilon \lim fx \,.=\Lambda$$

et, en opérant la Ths. par $\overline{z \varepsilon}$,

(ρ) $$\mathrm{Hp}(x) \,.\supset.\, (y + \mathrm{Q}) \frown \lim fx = \Lambda.$$

Analoguement la P30′ devient

(ρ') $$\mathrm{Hp}(x) \,.\supset.\, (y - \mathrm{Q}) \frown \lim fx = \Lambda.$$

Or, P20, $\lim fx = (\mathrm{q} \smile \iota \infty \smile \iota - \infty) \frown \lim fx$. L'ensemble des nombres réels q se décompose dans les nombres inférieurs, égaux, et supérieurs à y ; $\mathrm{q} = (y - \mathrm{Q}) \smile \iota y \smile (y + \mathrm{Q})$; donc

(σ) $$\mathrm{Hp}(x) \,.\supset.\, \lim fx = [\iota \infty \smile (y + \mathrm{Q}) \smile \iota y \smile (y - \mathrm{Q}) \smile \iota(-\infty)] \frown \lim fx.$$

Or la (ξ) dit que $\iota \infty \frown \lim fx = \Lambda$, et que $(\iota - \infty) \frown \lim fx = \Lambda$; les (ρ) et (ρ') donnent $(y + \mathrm{Q}) \frown \lim fx = \Lambda$, et $(y - \mathrm{Q}) \frown \lim fx = \Lambda$; donc la (σ) se transforme en :

(τ) $$\mathrm{Hp}(x) \,.\supset.\, \lim fx = \iota y \frown \lim fx,$$

qui, par l'identité logique $a = ab \,.=.\, a \supset b$ devient

$$(\upsilon) \qquad \mathrm{Hp}(\varkappa) \,.\supset.\, \lim fx \supset \iota y .$$

On peut écrire la $(\varkappa)$ sous la forme:

$$(\phi) \qquad \mathrm{Hp}(\varkappa) \,.\supset.\, \iota y \supset \lim fx .$$

Composons les (υ) et (ϕ); on a, en développant $\mathrm{Hp}(\varkappa)$:

$$(\chi) \qquad \mathrm{HpP31.}\ y = \mathrm{l}_1\mu' = \mathrm{l}'\mu_1 \,.\supset.\, \lim fx = \iota y .$$

Les propositions (ι) et (χ) ont la forme $ab \supset c \,.\, ac \supset b$, qui est réductible à la forme $a \supset . b = c$; en faisant cette combinaison des (ι) et (χ) on a la proposition à démontrer.

Nous voulons enfin réduire la proposition $\lim fx = \iota y$ à une autre forme, et précisement à la forme qu'on prend généralement par définition:

32. $\mathrm{HpP11.}\ y \varepsilon \mathrm{q} \,.\supset\therefore\cdot\, \lim fx = \iota y .$

$$= :: h \varepsilon \mathrm{Q} \,.\supset_h \therefore a \varepsilon \mathrm{q} : x \varepsilon u \,.\, x > a \,.\supset_x .\, \mathrm{mod}\,(fx - y) < h : \sim =_a \Lambda .$$

"Ayant u et f la signification connue, si y est un nombre fini, la condition nécessaire et suffisante pour que la limite de fx se réduise au nombre y, est que, pour toute valeur du nombre positif h, on puisse assigner un nombre a, tel que l'on ait $\mathrm{mod}\,(fx - y) < h$ pour toutes les valeurs de x, appartenant à l'ensemble u, et supérieures à a."

En effet, le théorème précedent transforme la proposition $\lim fx = \iota y$ en $y = \mathrm{l}_1\mu' = \mathrm{l}'\mu_1$:

$$(\alpha) \qquad \mathrm{HpP32.\ P31.} \supset : \lim fx = \iota y \,.=.\, y = \mathrm{l}_1\mu' = \mathrm{l}'\mu_1 .$$

Supposons donc $y = \mathrm{l}_1\mu' = \mathrm{l}'\mu_1$; prenons un nombre positif h; puisque y est la limite inférieure des μ', on peut déterminer un nombre m', appartenant à l'ensemble μ', et plus petit que $y + h$; et puisque y est la limite supérieure des μ_1, on peut déterminer un nombre m_1 de l'ensemble μ_1, plus grand que $y - h$:

$$(\beta) \qquad \mathrm{HpP32.}\ y = \mathrm{l}_1\mu' = \mathrm{l}'\mu_1 \,.\, h \varepsilon \mathrm{Q} \,.\supset: m' \varepsilon \mu' \,.\, m' < y + h \,.$$

$$\sim =_{m'} \Lambda : m_1 \varepsilon \mu_1 \,.\, m_1 > y - h \,.\sim =_{m_1} \Lambda .$$

Et étant m' et m_1 les nombres dont on vient de parler, par la définition de μ' et μ_1, on pourra déterminer un nombre a' tel que nulle valeur de $f[u \frown (a' + \mathrm{Q})]$

ne soit plus grande que m', et on pourra déterminer un nombre a_1 analogue pour la classe μ_1:

$$(\gamma)\quad \mathrm{Hp}(\beta) . m' \varepsilon \mu' . m' < y+h . m_1 \varepsilon \mu_1 . m_1 > y-h . \supset : a' \varepsilon \mathrm{q} . f[u \frown (a'+\mathrm{Q})] \frown (m'+\mathrm{Q}) \\ = \Lambda . \sim =_{a'} \Lambda : a_1 \varepsilon \mathrm{q} . f[u \frown (a_1+\mathrm{Q})] \frown (m_1 - \mathrm{Q}) = \Lambda . \sim =_{a_1} \Lambda .$$

Or, étant a' et a_1 les nombres dont on vient de parler, si l'on nomme a le plus grand des nombres a' et a_1, il est facile de voir que les valeurs de $f[u \frown (a+\mathrm{Q})]$ sont comprises entre $y - h$ et $y + h$:

$$(\delta)\quad \mathrm{Hp}(\gamma) . a' \varepsilon \mathrm{q} . f[u \frown (a'+\mathrm{Q})] \frown (m'+\mathrm{Q}) = \Lambda . a_1 \varepsilon \mathrm{q} . f[u \frown (a_1+\mathrm{Q})] \frown (m_1 - \mathrm{Q}) \\ = \Lambda . a = \max(a', a_1) . \supset . f[u \frown (a+\mathrm{Q})] \supset (y-h)\,^{-}(y+h) .$$

On prouve cette déduction par des transformations connues. Des Hp. on a $a \geq a'$, d'où $a + \mathrm{Q} \supset a' + \mathrm{Q}$, $f[u \frown (a + \mathrm{Q})] \supset f[u \frown (a' + \mathrm{Q})]$; mais $f[u \frown (a' + \mathrm{Q})] \frown (m' + \mathrm{Q}) = \Lambda$, par Hyp., donc $f[u \frown (a + \mathrm{Q})] \frown (m' + \mathrm{Q}) = \Lambda$, par la formule de logique $a \supset b . bc = \Lambda . \supset . ac = \Lambda$.

Analoguement on a $a \geq a_1$, d'où $f[u \frown (a + \mathrm{Q})] \supset f[u \frown (a_1 + \mathrm{Q})]$; mais $f[u \frown (a_1 + \mathrm{Q})] \frown (m_1 - \mathrm{Q}) = \Lambda$, donc $f[u \frown (a + \mathrm{Q})] \frown (m_1 - \mathrm{Q}) = \Lambda$.

On peut écrire ces deux relations sous la forme $f[u \frown (a + \mathrm{Q})] \supset m' - \mathrm{Q}_0$, $f[u \frown (a + \mathrm{Q})] \supset m_1 + \mathrm{Q}_0$; en les composant, on a:

$$f[u \frown (a + \mathrm{Q})] \supset (m' - \mathrm{Q}_0) \frown (m_1 + \mathrm{Q}_0);$$

mais m' et m_1 sont des nombres de l'intervalle $(y-h)\,^{-}(y+h)$; donc $(m' - \mathrm{Q}_0) \frown (m_1 + \mathrm{Q}_0) \supset (y-h)\,^{-}(y+h)$, d'où la Ths. de (δ).

On peut multiplier la Ths(δ) par $a \varepsilon \mathrm{q}$, qui est une conséquence des Hyp.:

$$(\varepsilon)\qquad \mathrm{Hp}(\delta) . \supset . a \varepsilon \mathrm{q} . f[u \frown (a + \mathrm{Q})] \supset (y-h)\,^{-}(y+h) .$$

Or de la Ths. de la (ε) "a est un nombre tel que ...." on déduit "on peut déterminer un nombre a tel que ....". Nous pouvons énoncer en général cette forme de déduction. Étant a_x une proposition contenant la lettre x, on a $a_x \supset . a_x \sim =_x \Lambda$ "si a_x est vraie, il y a des x qui satisfont à la condition a_x," dans notre cas:

$$(\zeta)\qquad \mathrm{Hp}(\delta) . \supset : a \varepsilon \mathrm{q} . f[u \frown (a + \mathrm{Q})] \supset (y-h)\,^{-}(y+h) . \sim =_a \Lambda .$$

Dans cette thèse la lettre a est apparente; elle ne contient que les lettres f, u, y, h; l'Hyp. contient encore les lettres a, a', a_1, m', m_1, que nous éliminons. On élimine a en supprimant sa définition $a = \max(a', a_1)$. On élimine

a' et a_1, en supprimant les propositions qui les contiennent, comme dit la (γ); on élimine m' et m_1 en supprimant aussi les propositions qui contiennent ces lettres, comme dit la (β); et l'on a:

(η) $\mathrm{HpP32}.\ y = \mathrm{l}_1\mu' = \mathrm{l}'\mu_1 . h \varepsilon \mathrm{Q} . \supset : a \varepsilon \mathrm{q} . f[u \frown (a+\mathrm{Q})] \supset (y-h)^{-}(y+h) . \sim =_a \Lambda .$

Exportons une partie de l'Hp.:

(θ) $\mathrm{HpP32}.\ y = \mathrm{l}_1\mu' = \mathrm{l}'\mu_1 . \supset \therefore h \varepsilon \mathrm{Q} . \supset_h : a \varepsilon \mathrm{q} . f[u \frown (a+\mathrm{Q})] \supset (y-h)^{-}(y+h) . \sim =_a \Lambda .$

Réciproquement, si h est une quantité positive et a un nombre tel que toutes les valeurs de $f[u \frown (a+\mathrm{Q})]$ appartiennent à l'intervalle de $y-h$ à $y+h$, il n'y a pas de valeur de $f[u \frown (a+\mathrm{Q})]$ supérieure à $y+h$, ni inférieure à $y-h$:

(ι) $\mathrm{HpP32}.\ h \varepsilon \mathrm{Q} . a \varepsilon \mathrm{q} . f[u \frown (a+\mathrm{Q})] \supset (y-h)^{-}(y+h) . \supset . f[u \frown (a+\mathrm{Q})$
$\frown (y+h+\mathrm{Q}) = \Lambda . f[u \frown (a+\mathrm{Q})] \frown (y-h-\mathrm{Q}) = \Lambda .$

Mais si $f[a \frown (a+\mathrm{Q})] \frown (y+h+\mathrm{Q}) = \Lambda$, $y+h$ est un nombre de l'ensemble μ'; analoguement $y-h$ est un μ_1:

$(\varkappa)$ $\mathrm{Hp}(\iota) . \supset . y+h \varepsilon \mu' . y-h \varepsilon \mu_1 .$

Donc $\mathrm{l}_1\mu' \leq y+h$, $\mathrm{l}'\mu_1 \geq y-h$; et en tenant compte de la P25 on a:

(λ) $\mathrm{HpP32}.\ h \varepsilon \mathrm{Q} . a \varepsilon \mathrm{q} . f[u \frown (a+\mathrm{Q})] \supset (y-h)^{-}(y+h) . \supset . y+h \geq \mathrm{l}_1\mu' \geq \mathrm{l}'\mu_1 \geq y-h .$

Éliminons a, qui figure seulement dans l'Hyp.:

(μ) $\mathrm{HpP32}.\ h \varepsilon \mathrm{Q} : a \varepsilon \mathrm{q} . f[u \frown (a+\mathrm{Q})] \supset (y-h)^{-}(y+h) . \sim =_a \Lambda : \supset . \mathrm{Ths}(\lambda) .$

"Si h est un nombre positif, et si l'on peut déterminer un nombre a tel que ...., alors les quantités $y+h$, $\mathrm{l}_1\mu'$, $\mathrm{l}'\mu_1$, $y-h$ forment une suite décroissante." De la (μ) on tire

(ν) $\mathrm{HpP32} \therefore h \varepsilon \mathrm{Q} . \supset_h : a \varepsilon \mathrm{q} . f[u \frown (a+\mathrm{Q})] \supset (y-h)^{-}(y+h) .$
$\sim =_a \Lambda \therefore \supset : h \varepsilon \mathrm{Q} . \supset_h . y+h \geq \mathrm{l}_1\mu' \geq \mathrm{l}'\mu_1 \geq y-h .$

On peut énoncer la loi avec laquelle on passe de la (μ) à la (ν). La (μ) a la forme $abc \supset d$, la (ν) a la forme $a . b \supset c . \supset . b \supset d$; la loi est donc:

$$abc \supset d . \supset : a . b \supset c . \supset . b \supset d .$$

On ne trouve pas cette formule logique dans le Formulaire; mais, en important toutes les Hyp. on la réduit à des formes qui y sont contenues. La (ν) signifie "si quelque soit le nombre positif h, on peut déterminer un nombre a tel que, ... , alors, quelque soit le nombre positif h, on a que . . . ."

Maintenant, de la Ths. (ν), "quelque soit h, on a $y+h \geq \mathrm{l}_1\mu' \geq \mathrm{l}'\mu_1 \geq y-h$," on déduit $y = \mathrm{l}_1\mu' = \mathrm{l}'\mu_1$; donc:

$$(\xi) \qquad \mathrm{Hp}(\nu) \supset . \, y = \mathrm{l}_1\mu' = \mathrm{l}'\mu_1 .$$

Les prop. (θ) et (ξ) ont la forme $ab \supset c \, . \, ac \supset b$; leur ensemble est donc réductible à la forme $a \supset . \, b = c$:

$$(\pi) \quad \mathrm{HpP32.} \supset :: y = \mathrm{l}_1\mu' = \mathrm{l}'\mu_1 .$$
$$= \therefore h \varepsilon \mathrm{Q} \, . \supset_h : a \varepsilon \mathrm{q} \, . \, f[u \frown (a+\mathrm{Q})] \supset (y-h)^{-}(y+h) \, . \sim =_a \Lambda .$$

Substituons ici à $y = \mathrm{l}_1\mu' = \mathrm{l}'\mu_1$ sa valeur $\lim fx = \iota y$, comme dit la (α). Au lieu de $f[u \frown (a+\mathrm{Q})] \frown (y-h)^{-}(y+h)$ on peut écrire $x \varepsilon u \, . \, x > a \, . \supset . \, \mathrm{mod}\,(fx-y) > h$; et la (π) se transforme dans la proposition à démontrer.

Maintenant on pourrait se proposer de développer la théorie des limites, et de généraliser les propositions connues, lorsqu'il y a une seule limite, en les rendant valables sans faire cette hypothèse. Par exemple le théorème de Cauchy:

$$f \varepsilon \mathrm{q\,f\,N} \, . \, \lim_{x,\, N,\, \infty} [f(x+1) - fx] \, \varepsilon \mathrm{q} \, . \supset . \lim \frac{fx}{x} = \lim \, [f(x+1) - fx] .$$

"Étant f une fonction réelle définie pour les valeurs entières de la variable, c'est-à-dire, étant $f1, f2, f3$, ... une suite de nombres, si la limite de $f(x+1) - fx$ a une valeur déterminée et finie, alors la limite de $\frac{fx}{x}$ coincide avec la limite précédente," se généralise en:

$$f \varepsilon \mathrm{q\,f\,N} \, . \supset . \lim \frac{fx}{x} \supset \mathrm{med} \lim \, [f(x+1) - fx] .$$

"Quelle que soit la suite $f1, f2$, ... , toute valeur limite de $\frac{fx}{x}$ est moyen entre les valeurs limites de la différence $f(x+1) - fx$."

Mais nous arrêtons ici notre exercice. On remarquera que la Logique mathématique représente avec le plus petit nombre de conventions toutes les propositions de mathématique, même celles très compliquées, dont la traduction en langage ordinaire serait fatiguante. Mais elle ne se réduit pas simplement à une écriture symbolique abrégée, à une espèce de tachygraphie; elle permet d'étudier les lois de ces signes, et les transformations des propositions. Nous avons expliqué les lois que nous avons rencontrées. D'abord cette analyse est longue; ensuite on voit que ce sont toujours les mêmes règles qui se présentent. On compose ou décompose des propositions, on importe, exporte, transporte des conditions, on élimine des lettres, on forme des syllogismes, etc. Seulement le syllogisme a été étudié par les anciens logiciens; on n'a découvert les autres règles qu'après l'introduction des symboles. Les deux objets de la logique mathématique, la formation d'une écriture symbolique, et l'étude des formes de transformations ou de raisonnement, sont étroitement liés. Nous terminerons avec les mots de Condillac.* "Tout l'art de raisonner se réduit à bien faire la langue de chaque science. Plus vous abrégerez votre discours, plus vos idées se rapprocheront; et plus elles seront rapprochées, plus il vous sera facile de les saisir sous tous leurs rapports."

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*La Logique, 1780, pag. 127 et 149.

Theorems in the Calculus of Enlargement.[*]

BY EMORY McCLINTOCK.

In my *Essay on the Calculus of Enlargement* (*Am. Journal of Mathematics*, II, 101–161), that calculus was described, from one point of view, as an extension of the Calculus of Finite Differences, comprising, as its most important branch, the Differential Calculus. I argued that the operation of Enlargement, indicated by

$$\mathrm{E}^h\phi x = \phi(x+h),$$

is simpler than that of Differentiation,

$$\mathrm{D}\phi x = \frac{\phi(x+h)-\phi x}{h}_{[h=0]};$$

that the two operations, E and D, are functions of each other; that whichever is defined last must be expressed in terms of the other; that D should therefore be defined in terms of E, namely, $\mathrm{D} = \log \mathrm{E}$; and that the theory of the functions of E, or Calculus of Enlargement, is a formal algebra, of which the theory of differentiation is that part which corresponds to the theory of logarithms in ordinary algebra. Spontaneous expressions of approval of these suggestions were sent to me by eminent mathematicians of different countries, and I cannot doubt that the ideas in question, being founded in reason, will in time find general acceptance.

In that Essay I gave incidentally (p. 146) several substitutes for Taylor's theorem, by which the coefficients were exhibited in the language of finite differences, or, as I prefer to say, of the calculus of enlargement, without reference to the theory of differentiation. My present purpose is to present another similar series, corresponding to Taylor's theorem, with a more direct proof, and with suitable illustrations, and afterwards to exhibit series corresponding to Lagrange's and Laplace's theorems. In doing this, several symbolic expansions of wide scope will be developed. Numbers preceded by the letter *E* will, wherever they occur, be understood to refer to the numbered equations of the Essay in Vol. II.

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[*] **Read before the American Mathematical Society, Aug. 14, 1894.**

Let $\mathrm{B} = \mathrm{E} - z$,

so that $\mathrm{B}\phi x = \phi(x+1) - z\phi x$,

and let $x^{m)} = x(x-1)(x-2)\ldots(x-m+1)$,

as in *E* 271. Then*

$$\begin{aligned} \mathrm{B}z^{x-m}x^{m)} &= z^{x+1-m}(x+1)^{m)} - z^{x+1-m}x^{m)} \\ &= z^{x+1-m}(x+1-x+m-1)\,x^{m-1)} \\ &= mz^{x-m+1}x^{m-1)}. \end{aligned}$$

Similarly,

$$\begin{aligned} \mathrm{B}^2 z^{x-m}x^{m)} &= m(m-1)\,z^{x-m+2}x^{m-2)}, \\ \mathrm{B}^n z^{x-m}x^{m)} &= m^{n)}z^{x-m+n}x^{m-n)} \\ \mathrm{B}^m z^{x-m}x^{m)} &= m!\,z^x. \end{aligned}$$

It will be observed that $\mathrm{B}^n z^{x-m}x^{m)} = 0$ when $n > m$. Let $\phi\mathrm{E}$ be any function of E which can be expressed in positive integral powers of B, say,

$$\phi\mathrm{E} = a_0 + a_1\mathrm{B} + a_2\mathrm{B}^2/2! + \ldots. \tag{1}$$

Then $$\phi\mathrm{E}z^{x-m}x^{m)} = a_0 z^{x-m}x^{m)} + ma_1 z^{x-m+1}x^{m-1)} + \ldots.$$

Let $x = 0$; then, since $0^{m)} = 0$ for all values of m greater than 0,

$$\phi\mathrm{E}_0 z^{0-m}0^{m)} = m^{m)}a_m z^{0-m+m}0^{m-m}/m! = a_m.$$

Hence, by substitution in (1),

$$\phi\mathrm{E} = \phi\mathrm{E}_0 z^0 + \phi\mathrm{E}_0 z^{0-1}0\mathrm{B} + \phi\mathrm{E}_0 z^{0-2}0^{2)}\mathrm{B}^2/2! + \ldots. \tag{2}$$

This theorem, doubtless new, may be illustrated in various ways. If, for example, it be applied to the problem of interpolation, we may take $z = 1$, $\phi\mathrm{E} = \mathrm{E}^n$, and we have, operating on fx, the well-known formula in finite differences, wherein $\Delta = \mathrm{E} - 1$,

$$f(x+n) = fx + n\Delta fx + n^{2)}\Delta^2 fx/2! + \ldots.$$

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* It will be understood that operations, functions of E, are indicated by symbols which in each case apply to all that follows in the same expression; while functional symbols, such as ϕ or f, apply only to the letter or bracket which they precede. For example, $\phi\mathrm{E}_z x^m f\mathrm{E}_z \psi x$ means that ψx is operated upon by $f\mathrm{E}_z$, the result multiplied by x^m, and the product operated upon by $\phi\mathrm{E}_z$. Continuity and intelligibility of results (equivalent in the case of series to convergence) are always presupposed.

Let $y = z + k$. We know, by E 112, that $\phi\mathrm{E}_0 y^0 = \phi y$, so that $\mathrm{B}_0^n y^0 = (\mathrm{E}_0 - z)^n y^0 = (y - z)^n = k^n$. If now we operate with both sides of (2) upon y^0 with respect to 0, we have, writing $z + k$ for y,

$$\phi(z + k) = \phi z + \phi\mathrm{E}_0 z^{0-1}0 . k + \phi\mathrm{E}_0 z^{0-2}0^{2)} . k^2/2\,! + \ldots . \tag{3}$$

Taylor's theorem declares the value of a_m in $\phi(z + k) = a_0 + a_1 k + a^2k^2/2\,! + \ldots .$ to be $\mathrm{D}_z^m \phi z$. This series (3) expresses the value of a_m by the use of the simpler symbol E, without reference to the operation of differentiation.*

In the use of (3) for the expansion of $\phi(z + k)$, the interpretation of the coefficients depends of course upon the form of the function denoted by ϕ. If, for example, $\phi z = z^n$,

$$\mathrm{E}^n z^{0-m} 0^{m)} = z^{n-m} n^{m)},$$

and we obtain the binomial theorem,

$$(z + k)^n = z^n + nz^{n-1}k + \ldots .$$

As another illustration of (3), let $\phi = \log$, and we have for the coefficient of $k^m/m\,!$

$$\begin{aligned}\log \mathrm{E}_0 z^{0-m} 0^{m)} &= 0^{m)} \log \mathrm{E}_0 z^{0-m} + z^{0-} \quad \log \mathrm{E}_0 0^{m)}, \text{ by } E \text{ 135,}\\ &= \log(1 + \Delta_0)\, 0^{m)} z^{-m}\\ &= [\Delta_0 - \tfrac{1}{2}\Delta_0^2 + \tfrac{1}{3}\Delta_0^3 - \ldots .]\, 0^{m)} z^{-m}\\ &= (-1)^{m-1}(m - 1)\,!\, z^{-m},\end{aligned}$$

since, by E 279, $\Delta^n 0^{m)} = 0$ except when $n = m$, in which case $\Delta^m 0^{m)} = m\,!$ Then (3) becomes

$$\log(z + k) = \log z + z^{-1}k - \tfrac{1}{2} z^{-2}k^2 + \ldots .$$

To interpret a trigonometrical expression, say $\sin \mathrm{E}_0 z^{0-m} 0^{m)}$, we use the analytical definition, say $\sin \mathrm{E} = \mathrm{E} - \mathrm{E}^3/3\,! + \ldots .$, and the result will vary according to the value of q in $m = 4p + q$. If, for instance, $m = 1$,

$$\begin{aligned}\sin \mathrm{E}_0 z^{0-1} 0 &= (\mathrm{E}_0 - \mathrm{E}_0^3/3\,! + \ldots .)\, z^{0-1}0\\ &= 1 - z^2/2\,! \quad + \ldots . = \cos z,\end{aligned}$$

and the same if $m = 5$:

$$\begin{aligned}\sin \mathrm{E}_0 z^{0-5} 0^{5)} &= (\mathrm{E}_0 - \mathrm{E}_0^3/3\,! + \ldots .)\, z^{0-5} 0^5\\ &= 5^{5)}/5\,! - z^2 7^{5)}/7\,! + \ldots .\\ &= 1 - z^2/2\,! + \ldots . = \cos z.\end{aligned}$$

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*In the earlier series, E 818, a_m is represented in the form $z^{-m}\phi(z\mathrm{E}_0)\,0^m$. The form in (8) is $\phi\mathrm{E}_0 z^{0-m} 0^{m)}$, so that $\phi(z\mathrm{E}_0)\,0^{m)}$ must be equivalent to $\phi\mathrm{E}_0 z\, 0^{m)}$. It is in fact easy to prove that $\phi(z\mathrm{E}_0)\,\psi 0 = \phi\mathrm{E}_0 z^0 \psi 0$. See E 98.

Again, if $m = 3$, $m = 7$, ..., we have the same result with the opposite sign, viz. $-\cos z$; if $m = 2$, $m = 6$, ..., $-\sin z$; and if $m = 4$, $m = 8$, ..., $+\sin z$. If, therefore, we write sin for ϕ in (3) we obtain

$$\sin(z + k) = \sin z + \cos z . k - \sin z . k^2/2! + \ldots .$$

In more complicated cases the coefficients are to be interpreted by observing general rules (equivalent to the usual rules for differentiation) which may be derived and proved, without reference to limits or differentials, by analytical methods alone, analogous to those laid down by Lagrange in his *Calcul des Fonctions*.

I would not be understood as suggesting this formula and other like formulæ mentioned in the previous paper referred to as improvements upon Taylor's theorem. My object is, however, something more than the mere exhibition of interesting novelties. This series (3) shows at a glance, what indeed has otherwise been abundantly proved, that there exists no barrier, no definite boundary, between the branches known as the Calculus of Finite Differences and the Differential Calculus. The Calculus of Enlargement, or algebra of the functions of E, comprises both those branches; the differential calculus, which relates to $\mathrm{D} = \log \mathrm{E}$, being that part of the symbolic algebra of the functions of E which corresponds to the theory of logarithms in ordinary algebra. Nor would I have thought it useful to present this new series (3) at this time at all, considering the other similar series given in the earlier paper, were it not that it happens to be naturally introductory to the presentation of wider and more important theorems.

The symbolic series (2) is but a special case of this:

$$\phi\mathrm{E} = \phi z + \phi\mathrm{E}_0 z^{0-h} 0 . \mathrm{A} + \phi\mathrm{E}_0 z^{0-2h} 0^{2]} . \mathrm{A}^2/2! + \ldots . \tag{4}$$

Here, as before, $\phi z = \phi\mathrm{E}_0 z^0$; and A represents $(\mathrm{E}^h - z^h)/h$, so that B is what A becomes when $h = 1$; also,

$$x^{m]} = x(x - h)(x - 2h) \ldots . (x - mh + h),$$

so that $x^{m)}$ is what $x^{m]}$ becomes when $h = 1$; so that, in short, (3) is what (4) becomes when $h = 1$. It is needless to recount the steps, exactly similar to those taken to prove (2), by which we may derive (4); we may note, however, that

$$\mathrm{A}_x z^{x-hm} x^{m]} = m z^{x-h(m-1)} x^{m-1]} .$$

Apart from the case (2) already considered, the most notable special case of (4) is that wherein $h = 0$. Let $h = 0$ and $z = e^u$; then since $(x^h - 1)/h = \log x$ when $h = 0$, $\mathrm{A} = \log \mathrm{E} - \log z = \mathrm{D} - u$, and (4) becomes

$$\phi\mathrm{E} = \phi(e^u) + \phi\mathrm{E}_0 e^{u0}0\,(\mathrm{D} - u) + \phi\mathrm{E}_0 e^{u0}0^2\,(\mathrm{D} - u)^2/2! + \ldots . \tag{5}$$

If $\phi\mathrm{E} = \log \mathrm{E} = \mathrm{D}$, this yields merely the identity $\mathrm{D} = u + \mathrm{D} - u$. If $\phi\mathrm{E} = \mathrm{E}^n$, and if we operate on fx and divide both sides by e^{un}, we derive a curious generalization of Taylor's theorem:

$$e^{-un}f(x + n) = fx + n\,(\mathrm{D} - u)\,fx + n^2\,(\mathrm{D} - u)^2 fx/2! + \ldots . \tag{6}$$

When $u = 0$, this becomes Taylor's theorem. If in (6), for example, $fx = e^{2ux}$, we have

$$e^{-un}e^{2u(x+n)} = e^{2ux} + nue^{2ux} + n^2u^2e^{2ux}/2! + \ldots .$$

If $y = e^{u+k}$, and if we operate with (5) upon y^0, remembering that $\phi\mathrm{E}_0 y^0 = \phi y$, we have this interesting result,

$$\phi(e^{u+k}) = \phi(e^u) + \phi\mathrm{E}_0 e^{u0}0\,.k + \phi\mathrm{E}_0 e^{u0}0^2.k^2/2! + \ldots . \tag{7}$$

or, substituting $\phi \log$ for ϕ,

$$\phi(u + k) = \phi u + \phi\mathrm{D}_0 e^{u0}0\,.k + \phi\mathrm{D}_0 e^{u0}0^2.k^2/2! + \ldots . \tag{8}$$

We have here still another substitute for Taylor's theorem, wherein, as will be observed, $\mathrm{D}_u^m \phi u = \phi\mathrm{D}_0 e^{u0}0^m$, a relation otherwise derivable at once from *E* 169, where $\phi\mathrm{D}_x\psi_x = \psi\mathrm{D}_0 e^{x0}\phi 0$. If in (7), as a special case, we put $u = 0$, we have Herschel's theorem,

$$\phi(e^k) = \phi 1 + \phi\mathrm{E}_0 0\,.k + \phi\mathrm{E}_0 0^2.k^2/2! + \ldots .$$

A more useful form of (7), for some purposes, may be

$$\phi(ze^k) = \phi z + \phi\mathrm{E}_0 z^0 0\,.k + \phi\mathrm{E}_0 z^0 0^2.k^2/2! + \ldots . \tag{9}$$

If for any reason we desire to ignore the operation of differentiation, we can determine the successive coefficients of (7) by writing $\phi_m u$ for $\phi\mathrm{E}_0 e^{u0}0^m$, and employing the relation

$$\phi_m \mathrm{E}_0 u^{0-1}0 = \phi\mathrm{E}_0 e^{u0}0^{m+1} = \phi_{m+1}u .$$

For, writing v for 0 in $\phi_m u$, to avoid using two kinds of zeros, we have $\phi_m u = \phi\mathrm{E}_v e^{uv}v^m$, and

$$\begin{aligned}
\phi_m \mathrm{E}_0 u^{0-1}0 &= \phi\mathrm{E}_v e^{v\mathrm{E}} v^m u^{0-1}0 \\
&= \phi\mathrm{E}_v v^m\,(1 + v\mathrm{E}_0 + \ldots\ )\,u^{0-1}0 \\
&= \phi\mathrm{E}_v v^m\,(v + v^2u + v^3u^2/2! + \ldots .) \\
&= \phi\mathrm{E}_v v^{m+1}e^{uv} \\
&= \phi\mathrm{E}_0 e^{u0}0^{m+1} = \phi_{m+1}u .
\end{aligned}$$

That is to say, having found the value of the m^{th} coefficient as a function of u, $\phi_m u$, we take the same function of E as an operator upon $u^{0-1}0$ to obtain the next coefficient.

The general principle which I have followed, in the earlier paper and in this, in the development of theorems of expansion, consists in assuming that expansion is practicable, and therefore that the form of the coefficients is all that is to be sought; and in finding a series of functions, $f_0x, f_1x, \ldots.$, and an operator P, such that $\mathrm{P}f_mx = mf_{m-1}x$, $\mathrm{P}f_0x = 0$; whence, as in the proof of (2), it follows that, for any function ϕ which can be expressed in positive integral powers,

$$\phi\mathrm{P} = \phi\mathrm{P}_0f_00 + \phi\mathrm{P}_0f_10 . \mathrm{P} + \phi\mathrm{P}_0f_20 . \mathrm{P}^2/2! + \ldots. \tag{10}$$

For the simplest series of functions, x^0, x^1, x^2, ..., the operator is D. Another series of functions is x^0, x, $x(x-h)$, $x(x-h)(x-2h)$, ..., commonly called factorials, and represented here by $x^{0]}$, $x^{1]}$, $x^{2]}$, ... The operator corresponding is E^h-1, and the resulting series is well known. In the earlier paper I extended the then existing theory by devising the more general form of factorial shown in *E* 265 and *E* 267,

$$x^{[m]} = x(x+amh-h)(x+amh-2h)\ldots.(x+amh-mh+h),$$

with the corresponding operator $(\mathrm{E}^{-ha+h}-\mathrm{E}^{-ha})/h$. The result was a very general operative series which included as special cases those just mentioned (in which respectively $h=0$ and $a=0$) and others more novel, such for example as those in which the operators are $1-\mathrm{E}^{-h}$, $\mathrm{E}^{\frac{1}{2}h}-\mathrm{E}^{-\frac{1}{2}h}$, DE^c. In the present paper the theorems (2) and (4) correspond to the operators $\mathrm{E}-z$ and $(\mathrm{E}^h-z^h)/h$ respectively, the former a special case of the latter. We have now to consider a still wider generalization involving the same principle of procedure.

Let $\mathrm{A} = (\mathrm{E}^h-z^h)/h$, as before, and let $x^{|m|} = x(\phi\mathrm{E}_x)^m x^{m-1|}$, where $x^{m-1|} = z^{x-mh}(x-h)(x-2h)\ldots.(x-mh+h)$. Also, $x^{|0|} = z^x$, $\mathrm{A}x^{|0|} = 0$. It is needed to prove that $\mathrm{A}x^{|m|} = m\phi\mathrm{E}_x x^{|m-1|}$.

We may in the first place prove that $\mathrm{A}x^{|m|} = m\phi\mathrm{E}_x x^{|m-1|}$ by analysis of the expression $(\phi\mathrm{E})^m$ contained in $x^{|m|}$, on the assumption that $\phi\mathrm{E}$ can be expressed in terms of A. Let $a_\alpha\mathrm{A}^\alpha$ and $a_\beta\mathrm{A}^\beta$ be any terms of $\phi\mathrm{E}$; then $a_\alpha\mathrm{A}^\alpha a_\beta\mathrm{A}^\beta$ will be a component of $(\phi\mathrm{E})^2$. When $\phi\mathrm{E}$ is used a third time as multiplier, let $a_\gamma\mathrm{A}^\gamma$ be any term of it; then $a_\alpha\mathrm{A}^\alpha a_\beta\mathrm{A}^\beta a_\gamma\mathrm{A}^\gamma$ will be a component of $(\phi\mathrm{E})^3$. In general, similarly, $a_\alpha a_\beta\ldots.a_\mu\mathrm{A}^{\alpha+\beta+\cdots+\mu}$ will be a component of $(\phi\mathrm{E})^m$, $a_\mu\mathrm{A}^\mu$ being any term of ϕE when used as a multiplier for the m^{th} time. Let us denote this component by $a_s\text{A}^s$. The corresponding component of $x^{|m|} = x(\phi\text{E})^m x^{m-1|}$ is therefore $a_s x\text{A}^s x^{m-1|}$. Let us denote this by comp $x^{|m|}$. Then

$$
\begin{aligned}
\text{A}\ \text{comp}\ x^{|m|} &= \text{A}a_s x(m-1)^{s)} x^{m-1-s|} \\
&= a_s(m-1)^{s)}(m-s)\,x\,.\,x^{m-s-2|*} \\
&= w(m-1)(m-s), \qquad (11)
\end{aligned}
$$

where $w = a_s(m-2)^{s-1)}x\,.\,x^{m-s-2|}$. Similarly, comp $(\phi\text{E})^{m-1} = a_s a_\mu^{-1}\text{A}^{s-\mu}$, and

$$
\begin{aligned}
\text{comp}\ m\phi\text{E}x^{|m-1|} &= \text{comp}\ m\phi\text{E}x(\phi\text{E})^{m-1}x^{m-2|} \\
&= ma_\mu\text{A}^\mu x a_s a_\mu^{-1}\text{A}^{s-\mu}x^{m-2|} \\
&= ma_s\text{A}^\mu x(m-2)^{s-\mu)}x^{m-s+\mu-2|*} \\
&= ma_s(m-2)^{s-\mu,}(m-s+\mu-1)^{\mu)}x\,.\,x^{m-s+2|} \\
&= wm(m-s+\mu-1). \qquad (12)
\end{aligned}
$$

The difference between (11) and (12) is $w(s-m\mu)$. This shows that no component of $\text{A}x^{|m|}$ is necessarily equal to the corresponding component of $m\phi\text{E}x^{|m-1|}$, except in the case where $\alpha = \beta = \ldots\ldots = \mu$, when $s = m\mu$, and the difference vanishes. In all other cases, however, we may so group the components as to find the sum of any group in $\text{A}x^{|m|}$ equal to that of the corresponding group in $m\phi\text{E}x^{|m-1|}$. As the first of the group, take the component already considered. As the next, take that component which is formed as first described, but with a cyclic interchange of factors, namely, with $a_\beta\text{A}^\beta$ in lieu of $a_\alpha\text{A}^\alpha$, $a_\gamma\text{A}^\gamma$ in lieu of $a_\beta\text{A}^\beta$, etc., ending with $a_\alpha\text{A}^\alpha$ in lieu of $a_\mu\text{A}^\mu$. The difference between the components in this case will be $w(s-m\alpha)$. In the next case, proceeding as before, the difference will be $w(s-m\beta)$, and so on, until when the group is completed the sum of the differences becomes $w(sm-ms)=0$. Since the sum of each group in $\text{A}x^{|m|}$ is equal to the sum of the corresponding group in $m\phi\text{E}x^{|m-1|}$, it follows that these two expressions are equivalent.

Another proof that $\text{A}x^{|m|} = m\phi\text{E}_x x^{|m-1|}$ depends on the assumption that ϕE can be expressed in powers of E, not necessarily positive or integral. We remark first that

$$(\phi\text{E})^2 xfx = 2\phi\text{E}x\phi\text{E}fx - x(\phi\text{E})^2 fx. \qquad (13)$$

For, the terms of ϕE being of the form $a\text{E}^\alpha$, $b\text{E}^\beta$, ..., each term of $(\phi\text{E})^2$ will be of the form $a^2\text{E}^{2\alpha}$ or $2ab\text{E}^{\alpha+\beta}$. As regards each term of the form $a^2\text{E}^{2\alpha}$, we have the identity

$$a^2\text{E}^{2\alpha}xfx = 2a\text{E}^\alpha xa\text{E}^\alpha fx - xa^2\text{E}^{2\alpha}fx,$$

---

* As will be seen by performing the operation denoted by A.

because the performance of the operations indicated produces the identity

$$a^2(x+2a)f(x+2a) = 2a^2(x+a)f(x+2a) - a^2xf(x+2a).$$

Similarly, as regards each term of the second form,

$$2ab\mathrm{E}^{\alpha+\beta}xfx = 2a\mathrm{E}^{\alpha}xb\mathrm{E}^{\beta}fx + 2b\mathrm{E}^{\beta}xa\mathrm{E}^{\alpha}fx - 2xab\mathrm{E}^{\alpha+\beta}fx.$$

By summing the terms of both forms we obtain (13). Let $fx = (\phi\mathrm{E})^{m-1}\psi x$; then (13) becomes

$$(\phi\mathrm{E})^2x(\phi\mathrm{E})^{m-1}\psi x = 2\phi\mathrm{E}x(\phi\mathrm{E})^m\psi x - x(\phi\mathrm{E})^{m+1}\psi x. \tag{14}$$

If the equation

$$(\phi\mathrm{E})^m x\psi x = m\phi\mathrm{E}x(\phi\mathrm{E})^{m-1}\psi x - (m-1)x(\phi\mathrm{E})^m\psi x \tag{15}$$

be true for any value of m, it is true for the value next higher, and so on for all higher values; for operating on (15) by $\phi\mathrm{E}$, and substituting for the first term of the second member its value from (14), we derive at once

$$(\phi\mathrm{E})^{m+1}x\psi x = (m+1)\phi\mathrm{E}x(\phi\mathrm{E})^m\psi x - mx(\phi\mathrm{E})^{m+1}\psi x.$$

But (15) is true, by (13), when $m=2$, and it is therefore true for all higher values. To go back now to $x^{|m|}$, its definition is,

$$x^{|m|} = x(\phi\mathrm{E}_x)^m z^{x-mh}(x-h)(x-2h)\ldots\ldots(x-mh+h).$$

If $\psi x = z^{x-mh}(x-h)\ldots\ldots(x-mh+2h)$, we have $x^{|m|} = x(\phi\mathrm{E})^m(x-mh+h)\psi x$. If we operate upon this with $\mathrm{A} = (\mathrm{E}^h - z^h)/h$, we have

$$\begin{aligned} \mathrm{A}x^{|m|} &= [(x+h)(\phi\mathrm{E})^m x\psi x - x(\phi\mathrm{E})^m(x-mh+h)\psi x]/h \\ &= (\phi\mathrm{E})^m x\psi x + (m-1)x(\phi\mathrm{E})^m\psi x \\ &= m\phi\mathrm{E}x(\phi\mathrm{E})^{m-1}\psi x, \text{ by (15)}, \\ &= m\phi\mathrm{E}x^{|m-1|}. \end{aligned} \tag{16}$$

Let $\mathrm{H} = \mathrm{A}(\phi\mathrm{E})^{-1} = (\phi\mathrm{E})^{-1}\mathrm{A}$, since all functions of E are commutative; then, performing the operation $(\phi\mathrm{E})^{-1}$ upon both sides of (16), we find that

$$\begin{aligned} \mathrm{H}x^{|m|} &= mx^{|m-1|}, \\ \mathrm{H}^n x^{|m|} &= m^{n)}x^{|m-n|}, \\ \mathrm{H}^m x^{|m|} &= m^{m)}x^{|0|} = m^{m)}z^x, \\ \mathrm{H}^{m+1}x^{|m|} &= 0. \end{aligned}$$

Let $f\mathrm{E}$ be any function of E which can be expressed in positive integral powers of H, say

$$f\mathrm{E} = a_0 + a_1\mathrm{H} + a_2\mathrm{H}^2/2! + \ldots\ldots \tag{17}$$

Then

$$f\mathrm{E}x^{|m|} = a_0x^{|m|} + a_1mx^{|m-1|} + a_2m^{2)}x^{|m-2|}/2! + \ldots\ldots,$$

whence, if $x=0$, $\qquad f\mathrm{E}0^{|m|}=a_m$.

Substituting this for all values of m in (17), and observing that $f\mathrm{E}0^{|0|}=f\mathrm{E}z^0=fz$, we obtain finally this general symbolic theorem:

$$f\mathrm{E}=fz+f\mathrm{E}_0 0^{|1|}\mathrm{H}+f\mathrm{E}_0 0^{|2|}\mathrm{H}^2/2\,!+\ldots\,, \tag{18}$$

where $\mathrm{H}=(\mathrm{E}^h-z^h)/h\phi\mathrm{E}$, and $x^{|m|}=x\,(\phi\mathrm{E}_x)^m z^{x-mh}(x-h)(x-2h)\ldots(x-mh+h)$. This comprehensive expression includes as special cases all the series already mentioned in this and the earlier paper as well as those yet to be given. If $\phi\mathrm{E}=\mathrm{E}^{ah}$ and if $z=1$, we have from (18) the "factorial theorem" (E 291) which was presented and extensively discussed in the former paper, where many deductions were made from it. A great number of special cases and applications might be deduced from the wider formula (18), but I shall confine attention at present to a few of the more notable cases.

Let $x=(y^h-z^h)/h\phi y$, so that x is the same function of y as H is of E; then, operating with (18) upon y^0 with respect to 0, and remembering that $f\mathrm{E}_0 y^0=fy$, we have the following general series for the expansion of a function of y in terms of x, when $y^h=z^h+xh\phi y$:

$$fy=fz+f\mathrm{E}_0 0^{|1|}.x+f\mathrm{E}_0 0^{|2|}.x^2/2\,!+f\mathrm{E}_0 0^{|3|}.x^3/3\,!+\ldots. \tag{19}$$

This looks like Lagrange's and Laplace's series, yet it is new, simple and useful, in and of itself. I find no indication that any one hitherto has attempted to expand in terms of x, by either of the series named, a function of y, when $y^h=z^h+xh\phi y$. If we put $y^h=u$, and expand $f(u^{\frac{1}{h}})$ by either series, we find that

$$\begin{aligned}fy=fz+z^{1-h}\phi zf'z.x+z^{1-h}\frac{d}{dz}\left[z^{1-h}(\phi x)^2f'z\right].x^2/2\,!&\\+\left(z^{1-h}\frac{d}{dz}\right)^2\left[z^{1-h}(\phi z)^3f'z\right].x^3/3\,!+\ldots.&\end{aligned} \tag{20}$$

This series, also new, is identical with (19) except in form, and may be preferred by those who are not acquainted with, or who find a difficulty in the use of, the notation of the calculus of finite differences. If $fy=y$, we have as special cases of (19) and (20) the following new and highly important developments, identical except in form:

$$y=z+z^{1-h}\phi z.x+\mathrm{E}_0(\phi\mathrm{E}_0)^2z^{0-2h}(0-h).x^2/2\,!+\ldots\,, \tag{21}$$

$$y=z+z^{1-h}\phi z.x+z^{1-h}\frac{d}{dz}\left[z^{1-h}(\phi z)^2\right].x^2/2\,!+\ldots. \tag{22}$$

The manner of employing this biform expression in determining approximately the roots of any equation, $y^h = z^h + hx\phi y$, is shown in the succeeding paper, "On a Method for Calculating Simultaneously all the Roots of an Equation."

If, in (20), $h = 1$, we have Lagrange's well-known theorem, where $y = z + x\phi y$,

$$fy = fz + \phi z f'z.x + \frac{d}{dz}[(\phi z)^2 f'z].x^2/2! \ldots., \tag{23}$$

corresponding to which we have, from (19),

$$fy = fz + f\mathrm{E}_0 0 \phi \mathrm{E} z^{0-1}.x + f\mathrm{E}_0 0\,(\phi \mathrm{E}_0)^2 z^{0-2}(0-1).x^2/2! + \ldots., \tag{24}$$

in which the coefficients have the same value as in (23), but are expressed without reference to the operation of differentiation. If $\phi \mathrm{E} = \mathrm{E}^0 = 1$, $y = z + x$, and (24) becomes reduced, as a special case, to (3), in which Taylor's theorem is replaced by one in which the coefficients are expressed without reference to differentiation. To obtain a corresponding expression in lieu of Laplace's theorem, which expands fu in terms of x, where $u = \psi(z + x\phi u)$, let $u = \psi y$, and from (24) we have

$$fu = f\psi z + f\psi \mathrm{E}_0 0 \phi\psi \mathrm{E} z^{0-1}.x + f\psi \mathrm{E}_0 0\,(\phi\psi \mathrm{E}_0)^2 z^{0-2}(0-1).x^2/2! \ldots., \tag{25}$$

Laplace's theorem being

$$fu = f\psi z + \phi\psi z \frac{d}{dz} f\psi z.x + \frac{d}{dz}\left[(\phi\psi z)^2 \frac{d}{dz} f\psi z\right].x^2/2! + \ldots.$$

Another new series, akin to (24), may be derived from (19) by putting $h = 0$, in which case $x\phi y = \frac{y^h - z^h}{h}_{[h=0]} = \log y - \log z$. That is to say, assuming $z = e^u$, the relation involved is $\log y = u + x\phi y$; and the series is

$$fy = f(e^u) + f\mathrm{E}_0 0 \phi \mathrm{E}_0 e^{u0}.x + f\mathrm{E}_0 0\,(\phi \mathrm{E}_0)^2 e^{u0} 0.x^2/2! + \ldots., \tag{26}$$

the general term being $f\mathrm{E}_0 0\,(\phi \mathrm{E}_0)^m e^{u0} 0^{m-1}.x^m/m!$. If, as a very simple special case, we put $u = 0$ and $\phi y = 1$, we have Herschel's theorem. If $\log y = v$, and if $\phi y = \psi \log y = \psi v$, we derive from (26), writing $f\log$ for f, still another important result,

$$fv = fu + f\mathrm{D}_0 0 \psi \mathrm{D}_0 e^{u0}.x + f\mathrm{D}_0 0\,(\psi \mathrm{D}_0)^2 e^{u0} 0.x^2/2! + \ldots. \tag{27}$$

Here $v = u + x\psi v$, and $\mathrm{D}_0 = \log \mathrm{E}_0 = \frac{d}{dx}_{[x=0]}$. Comparing this with Lagrange's theorem (23) and the corresponding series (24), we see that the coefficients of all three must be identical in value. We have here, in fact, another form for the expansion effected by Lagrange's theorem, a form which will be found entirely analogous in principle to the "secondary form of Maclaurin's theorem" given by Boole in the second chapter of his *Finite Differences*:

$$fv = f0 + f\mathrm{D}_0 0\,.\,v + f\mathrm{D}0^2.\,v^2/2\,! + \ldots .$$

To reduce (27) to this form we have merely to put $u = 0$, $\psi v = 1$, and therefore $x = v$. If, without going so far, we put $\psi v = 1$, the general series (27) assumes the restricted form given above in (8).

Lest the reader suppose that the theorems here developed are not practically available, in comparison with those of Lagrange and Laplace, I add one or two examples. Let it be required to expand y^n in terms of x when $y = ze^{xy^k}$. Referring to (26), we have $\phi y = y^k$, so that the general (m^{th}) term of the expansion desired is

$$\mathrm{E}_0^n 0 \mathrm{E}_0^{km} z^0 0^{m-1}.\,x^m/m\,! = n\mathrm{E}_0^{n+km} z^0 0^{m-1}.\,x^m/m\,! = nz^{n+km}(n+km)^{m-1}.\,x^m/m\,!.$$

Again, to expand y^n when $y = z + xy^k \log y$, we shall have from (24), as the m^{th} term,

$$\begin{aligned}&\mathrm{E}_0^n 0 \mathrm{D}_0^m \mathrm{E}_0^{km} z^{0-m}(0-1)(0-2)\ldots.(0-m+1).\,x^m/m\,!\\ &\quad = n\left(\frac{d}{d0}\right)^m z^{n+km-m+0}(0+n+km-1)\ldots.(0+n+km-m+1)\,x^m/m\,!.\end{aligned}$$

Apart from that in which $h = 1$, the chief symbolic special case of (18) is that in which $h = 0$. In this case $\mathrm{H} = (\mathrm{D} - u)/\phi\mathrm{E}$, where $\mathrm{D} = \log \mathrm{E} = \dfrac{d}{dy}$ (assuming the operations to be with respect to y), and $u = \log z$; also $x^{|m|} = x\,(\phi\mathrm{E}_x)^m e^{ux} x^{m-1}$. But in fact (18) is so sweepingly comprehensive that to dwell longer on special cases would only weary any reader not sufficiently interested to seek out the cases for himself.

It has been remarked that the coefficients of (24) and (27) must have the same value as those of Lagrange's theorem (23); that is, that

$$\left(\frac{d}{dz}\right)^{m-1}[(\phi z)^m f'z] = f\mathrm{E}_0 0\,(\phi\mathrm{E}_0)^m z^{0-m}(0-1)(0-2)\ldots.(0-m+1) \tag{28}$$

$$= f\mathrm{D}_0 0\,(\phi\mathrm{D}_0)^m e^{z0} 0^{m-1}. \tag{29}$$

These relations may, assuming f and ϕ developable by Maclaurin's theorem, be proved independently as follows. Since $f\mathrm{E}_0 x^0 = fx$, and $\frac{d}{dz} fz = f\mathrm{E}_0 z^{0-1}0$,

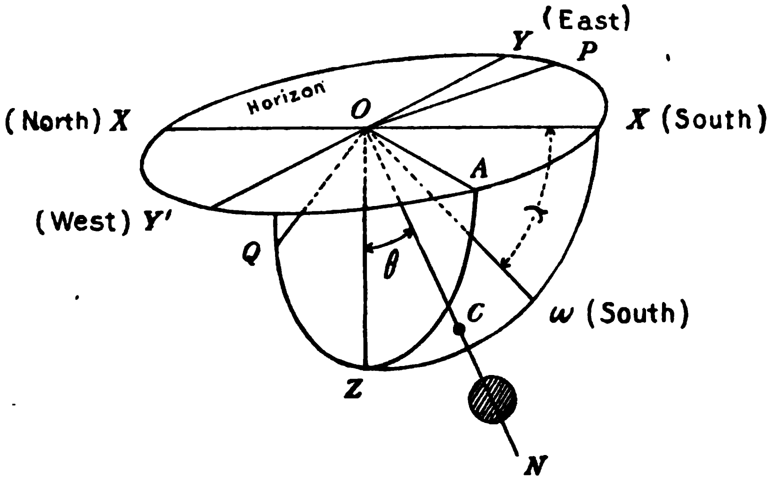
$$
\begin{aligned}
(\phi z)^m f'z &= (f\mathrm{E}_0 z^{0-1}0) \,.\, (\phi \mathrm{E}_0)^m z^0 \\
&= f\mathrm{E}_0 0 \, (\phi \mathrm{E}_0)^m z^{0-1}; \qquad (30)
\end{aligned}
$$

for, if the general term in $f\mathrm{E}$ is $a\mathrm{E}^\alpha$ and that in $(\phi\mathrm{E})^m$ is $b\mathrm{E}^\beta$, the general term of the second member is $(a\mathrm{E}^\alpha z^{0-1}0) \,.\, b\mathrm{E}^\beta z^0 = ab\alpha z^{\alpha-1+\beta}$, and this is the general term of the final form. If we differentiate both sides of (30) $m - 1$ times with respect to z, we obtain (28). Again, the general term of $f\mathrm{D}_0 0 \, (\phi \mathrm{D}_0)^m e^{z0}$ is $a\mathrm{D}_0^\alpha 0 b \mathrm{D}_0^\beta e^{z0} = a\alpha\mathrm{D}_0^{\alpha-1} b\mathrm{D}_0^\beta e^{z0} = a\alpha b e^{z0} z^{\alpha+\beta-1} = a\alpha b z^{\alpha+\beta-1}$, so that (29) may be proved similarly.

On Foucault's Pendulum.

By A. S. Chessin.

In this paper we shall consider the motion of a physical pendulum on the surface of the earth, taking into account the rotation of the earth about its axis. The initial velocity of the pendulum relatively to the earth will be supposed equal to zero, as in the famous experiment of Foucault with his practically mathematical pendulum. The name of "Foucault's pendulum" is therefore retained, although oscillations of any finite amplitude will be considered.*



Let O be a point on the surface of the earth; OXZ the meridian of this point; $O\omega$ the direction of the axis of the earth towards the south. Then $\angle XO\omega$ is the latitude λ of the place, QX being the intersection of the meridian plane with the horizon XOY, directed again towards the south; OY in the plane of the horizon 90° from OX and towards the east; OZ in the direction of the force of gravity. Farther, let ON be a physical pendulum, θ the angle its axis makes with OZ, the pendulum being a body of rotation. Let OP, OQ, ON be a system of three principal axes of inertia about the point O, the axis ON coinciding with the geometrical axis of the pendulum, OQ under 90° to ON in the plane $OANZ$

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* As is well known, only very small oscillations were given to Foucault's pendulum.

(the plane which would be the plane of oscillations of the pendulum but for the disturbance due to the rotation of the earth); OP, perpendicular to this plane in the sense shown on the figure. Let farther C be the centre of inertia of the pendulum and $OC = l$; M be the mass of the body; A, A, C be the three principal moments of inertia of the pendulum about the point O; $\gamma_1, \gamma_2, \gamma_3$ the angles which the axes OP, OQ, ON make with the direction of the axis (ω) of the earth. Finally, let θ, ϕ, ψ be the three Eulerian angles which define the position of the pendulum relative to the earth, i. e. $\theta = \angle NOZ$; $\phi = \angle POX$; $\psi =$ angle of a determinate plane passing through ON and fixed in the body, with the plane OZN. Then we shall have

$$\begin{aligned} \cos\gamma_1 &= \cos\lambda\cos\phi, \\ \cos\gamma_2 &= \sin\lambda\sin\theta - \cos\lambda\cos\theta\sin\phi, \\ \cos\gamma_3 &= \sin\lambda\cos\theta + \cos\lambda\sin\theta\sin\phi. \end{aligned} \tag{1}$$

We shall make use of Bour's differential equations for the relative motion

$$\frac{d}{dt}\left(\frac{\partial T_2}{\partial q'_k}\right) - \frac{\partial T_2}{\partial q_k} = \frac{\partial(U+K)}{\partial q_k},$$

where T_2 is the kinetic energy of the *absolute* rotation, $U + K$ is the potential of the force of *gravity* (and not of attraction)* in the case of the earth. Hence

$$U + K = Mgl\cos\theta, \tag{2}$$

$$T_2 = \tfrac{1}{2}\{A[(\theta' + \omega\cos\gamma_1)^2 + (\phi'\sin\theta + \omega\cos\gamma_2)^2] + C(r + \omega\cos\gamma_3)^2\}, \tag{3}$$

where $$r = \psi' + \phi'\cos\theta,$$

and ω is the angular velocity of the rotation of the earth. Bour's equation gives first ($q_k = \psi$),

$$\frac{d}{dt}\left(\frac{\partial T_2}{\partial \psi'}\right) = 0,$$

as neither T_2 nor $U + K$ contains the variable ψ, and thus we have a first integral

$$r + \omega\cos\gamma_3 = \psi' + \phi'\cos\theta + \omega\cos\gamma_3 = c_1.$$

We have supposed the initial velocity of the pendulum equal to zero, hence $c_1 = \omega\cos\gamma_{30}$ and

$$r + \omega\cos\gamma_3 = \omega\cos\gamma_{30}, \tag{5}$$

---

* U is the potential of the force of attraction.

where $\cos\gamma_{30}$ means the substitution of the initial values θ_0, ϕ_0 into the expression of $\cos\gamma_3$.

Next take $q_k = \phi$: $$\frac{d}{dt}\left(\frac{\partial T_2}{\partial\phi'}\right) - \frac{\partial T_2}{\partial\phi} = 0,$$

or

$$\frac{d}{dt}\left[A(\phi'\sin\theta + \omega\cos\gamma_3)\sin\theta + C\omega\cos\gamma_{30}\cos\theta\right] = -A\omega\cos\lambda\sin\theta\left[\theta' + \phi'\cos\theta\cos\phi\right],$$

leaving off the terms of the order of ω^2.* After a slight transformation this equation becomes

$$\frac{d}{dt}\left[A\sin^2\theta(\phi' + \omega\sin\lambda) + C\omega\cos\gamma_{30}\cos\theta\right] = -2A\omega\cos\lambda\sin^2\theta\sin\phi\theta'. \qquad (6)$$

If we neglect terms of the order of ω^2 and higher, we may substitute in the right-hand member of (6) the value of $\sin^2\theta\sin\phi.\theta'$ calculated in the supposition of the immobility of the earth, because this expression has the factor ω. Let ϑ be the angle formed by the axis of the pendulum with OZ in this hypothesis. Then we may instead of $\sin^2\theta\sin\phi.\theta'$ in (6) substitute $\sin\phi_0\sin^2\vartheta.\vartheta'$. Integrating the equation after that and putting for brevity

$$f_1(\theta) = \sin^2\theta_0 - \sin^2\theta + \frac{C}{A}\cos\theta_0(\cos\theta_0 - \cos\theta), \qquad (7)_1$$

$$f_2(\theta) = \frac{C}{A}\sin\theta_0(\cos\theta_0 - \cos\theta), \qquad (7)_2$$

$$f_3(\vartheta) = \theta_0 - \sin\theta_0\cos\theta_0 - (\vartheta - \sin\vartheta\cos\vartheta) \qquad (7)_3$$

we shall find

$$\sin\theta.\phi' = \omega\frac{f_1(\theta)\sin\lambda + [f_2(\theta) + f_3(\vartheta)]\cos\lambda\sin\phi_0}{\sin\theta}. \qquad (8)$$

This value of $\sin\theta.\phi'$ we now substitute in the third and last integral, that of the kinetic energy in the relative motion:

$$A(\theta'^2 + \sin^2\theta.\phi'^2) = 2Mgl(\cos\theta - \cos\theta_0). \qquad (9)$$

Now, the expression (8) shows that $\sin^2\theta.\phi'^2$ involves the factor ω^2. But this term *cannot be neglected.* This exception is due to the fact that in the right-hand member of (8) $\sin\theta$ appears in the denominator. We do not know *a priori* what the minimum value of θ may be. It may perhaps be zero, like the minimum value of ϑ (i. e. like in the ordinary pendulum), in which case the right-hand member of (8) would become infinite; or it may be of the order of ω, in which case the same expression would have a finite value. It follows from this remark that the term $\frac{\omega^2}{\sin^2\theta}\{f_1(\theta)\sin\lambda + [f_2(\theta) + f_3(\vartheta)]\cos\lambda\sin\phi_0\}^2$ *cannot be neglected a priori whatever be the desired approximation.* There are, however, some terms in the expression (8) which may be dropped, namely, $-\sin^2\theta$ in $(7)_1$, $(\vartheta - \sin\vartheta\cos\vartheta)$ in $(7)_3$. This is obvious. Hence, we must substitute for $\sin^2\theta . \phi'^2$ into the left-hand member of (9) the expression $\frac{\omega^2 f^2(\theta)}{\sin^2\theta}$, where

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*In this problem, like in all other problems of motions on the surface of the earth, it is useless to keep terms of the order of ω^2, if the force of gravity is considered as constant.

$$f(\theta) = \sin\lambda\left[\sin^2\theta_0 + \frac{C}{A}\cos\theta_0(\cos\theta_0 - \cos\theta)\right]$$
$$+ \cos\lambda\sin\phi_0\left[\frac{C}{A}\sin\theta_0(\cos\theta_0 - \cos\theta) + \theta_0 - \sin\theta_0\cos\theta_0\right]. \quad (10)$$

We then obtain the equation

$$\left(\frac{d\cos\theta}{dt}\right)^2 = \frac{2Mgl}{A}\sin^2\theta(\cos\theta - \cos\theta_0) - \omega^2 f^2(\theta), \quad (11)$$

the integration of which gives

$$\cos\theta = \cos\theta_0\,\mathrm{sn}^2\mu(t + T) + (1 - \tfrac{1}{2}\varepsilon^2)\,\mathrm{cn}^2\mu(t + T),$$

where

$$\left.\begin{aligned} \mu &= \sqrt{\frac{Mgl}{A}}, \\ T &= \sqrt{\frac{A}{Mgl}}\int_0^{\frac{\pi}{2}}\frac{d\phi}{\sqrt{1 - k^2\sin^2\phi}}, \\ k &= \sin\frac{\theta_0}{2}, \end{aligned}\right\} \quad (12)$$

$$\varepsilon = \omega\frac{f(0)}{2k\mu}. \quad (13)$$

We may write $\cos\varepsilon$ instead of $(1 - \frac{1}{2}\varepsilon^2)$, then

$$\cos\theta = \cos\theta_0\,\mathrm{sn}^2\mu(t + T) + \cos\varepsilon\,\mathrm{cn}^2\mu(t + T). \quad (14)$$

To determine ϕ, the equation (8) gives

$$\phi' = -\omega\sin\lambda + \omega\frac{f(\theta)}{\sin^2\theta} - \omega\cos\lambda\sin\phi_0\frac{\vartheta - \sin\vartheta\cos\vartheta}{\sin^2\theta}. \quad (15)$$

The expression of ϑ in function of time is found from (14), where we only need to put $\varepsilon = 0$. Then $\sin\frac{\vartheta}{2} = \sin\frac{\theta_0}{2}\operatorname{sn}\mu(t+T)$ and $\cos\frac{\vartheta}{2} = \operatorname{dn}\mu(t+T)$. Thus $\frac{\vartheta - \sin\vartheta\cos\vartheta}{\sin^2\theta}$ is readily obtained in function of t. Let us put

$$\phi(t) = \frac{\vartheta - \sin\vartheta\cos\vartheta}{\sin^2\theta} \text{ and} \int_0^t \phi(t)\,dt = \Phi(t). \tag{16}$$

Then, as easily seen, $\Phi(t)$ is a periodical function of the period $2T$. We have furthermore

$$\frac{f(\theta)}{\sin^2\theta} = \frac{\frac{1}{2}f(0)}{1-\cos\theta} + \frac{\frac{1}{2}f(\pi)}{1+\cos\theta},$$

$$1 - \cos\theta = -2k^2(\operatorname{sn}^2\eta i - \operatorname{sn}^2\mu(t+T)),$$

$$1 + \cos\theta = \quad 2\operatorname{dn}^2\mu(t+T),$$

where we have put $\eta = \frac{\varepsilon}{2k}$; and very simple integrations give

$$\int_0^t \frac{dt}{\operatorname{sn}^2\eta i - \operatorname{sn}^2\mu(t+T)} = \frac{i}{2\mu\eta}\lg\frac{\Theta_2(\mu t - \eta i)}{\Theta_2(\mu t + \eta i)} - \zeta t,$$

$$\int_0^t \frac{dt}{\operatorname{dn}^2\mu(t+T)} \qquad = \frac{1}{k'^2}(1-\zeta)t + \frac{1}{\mu k'^2}\frac{\Theta'(\mu t)}{\Theta(\mu t)}.$$

A transformation, which will be easily verified by the reader, gives

$$\frac{1}{2i}\lg\frac{\Theta_2(\mu t - \eta i)}{\Theta_2(\mu t + \eta i)} = \operatorname{tn}^{-1}\left[\eta\frac{\Theta_2'(\mu t)}{\Theta_2(\mu t)}\right].$$

In these formulas, as is usual,

$$\zeta = \frac{\Theta''(0)}{\Theta(0)} = \frac{1}{\mu T}\int_0^{\mu T} k^2\operatorname{sn}^2 x\,dx. \tag{17}$$

Substituting the above formulas into (15), integrating and putting for the sake of brevity

$$N = -\sin\lambda + \tfrac{1}{4}\left[\frac{f(0)}{k^2}\zeta + \frac{f(\pi)}{k'^2}(1-\zeta)\right], \tag{18}$$

$$F(t) = \frac{f(\pi)}{4k'^2\mu}\frac{\Theta'(\mu t)}{\Theta(\mu t)} - \cos\lambda\sin\phi_0\Phi(t), \tag{19}$$

we shall find

$$\phi = \phi_0 + \omega N t + \mathrm{tn}^{-1}\left[\eta \frac{\Theta_2'(\mu t)}{\Theta_2(\mu t)}\right] + \omega F(t). \tag{20}$$

If we put furthermore

$$\phi_1 = \mathrm{tn}^{-1}\left[\eta \frac{\Theta_2'(\mu t)}{\Theta_2(\mu t)}\right] + \omega F(t),$$

or

$$\mathrm{tg}\,[\phi_1 - \omega F(t)] = \eta \frac{\Theta_2'(\mu t)}{\Theta_2(\mu t)}, \tag{21}$$

the motion of the axis of the pendulum may be represented in the following way:

1. Equation (14) shows that *the axis of the pendulum oscillates between the positions* $\theta = \theta_0$ *and* $\theta = \varepsilon$, *never passing through the vertical.*

2. *The equations* (14) *and* (21) *represent a closed cone with a plane of symmetry which would be the plane of oscillations but for the disturbance due to the rotation of the earth.*

3. *If we rotate the cone just defined, about the vertical OZ with the constant angular velocity* ωN; *the combined motion: of the cone about OZ, and of the axis of the pendulum on this cone, will represent the motion of the axis of the pendulum relatively to the earth.*

4. *The rotation of the cone defined above may take place towards the west or towards the east, or the cone may be fixed relatively to the earth, according to whether* N *is less than, greater than, or equal to zero.*[*]

In fact, formula (18) can be written in the following way:

$$N = -n_1 \sin\lambda + n_2 \cos\lambda \sin\phi_0, \tag{22}$$

$$n_1 = \zeta k^2 + (1-\zeta) k'^2 - \frac{C}{2A}(1-2\zeta)\cos\theta_0, \tag{23}$$

$${}_2 = \frac{\theta_0 - \sin\theta_0\cos\theta_0}{\sin^2\theta_0}[\zeta k'^2 + (1-\zeta)k^2] + \frac{C}{2A}(1-2\zeta)\sin\theta_0, \tag{24}$$

and it is easily verified that *both* n_1 *and* n_2 *have always positive values.* Hence N is $\gtreqless 0$ according as

$$n_2 \sin\phi_0 \gtreqless n_1\,\mathrm{tg}\,\lambda. \tag{25}$$

It follows from this discussion that *the rotation of the cone* defined above, *at the same latitude, depends* 1) *on the amplitude of the oscillations*; 2) *on the construction*

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[*] See Comte de Sparre: Sur le mouvement du pendule conique à la surface de la terre. (Thèse de Doctorat.)

of the pendulum, and 3) *on its initial orientation.* The dependence on the amplitude and the orientation has been indicated first by Count de Sparre in his doctor thesis, in which, however, only the motion of a *mathematical* pendulum was considered. The influence of the construction of the pendulum on the results, as seen from the formulas developed in this paper, offers a greater field for experiments which would be very interesting.

Formula (22) shows that, *ceteris paribus, the absolute value of N is maximum or minimum, if the pendulum is started in the plane of the meridian; maximum if initially deviated towards the north; minimum if towards the south.*

$$|N|_{\text{max.}} = n_1 \sin\lambda + n_2 \cos\lambda,$$
$$|N|_{\text{min.}} = |n_1 \sin\lambda - n_2 \cos\lambda|.$$

In order to complete the solution it remains to determine the angle ψ in function of t. This is done very easily, but the limits of this paper do not allow the reproduction of the calculation here.

To conclude, let us see what the above formulas become in the case of very small amplitudes of oscillations, neglecting powers of θ_0 higher than the second.

Formula (21) becomes

$$\operatorname{tg}\phi_1 = \eta \frac{\Theta_2'(\mu t)}{\Theta_2(\mu t)} = -\frac{\varepsilon}{\theta_0} \operatorname{tg}\mu t, \tag{21$'$}$$

and instead of the equation (14) we may write

$$\theta^2 = \theta_0^2 \cos^2\mu t + \varepsilon^2 \sin^2\mu t. \tag{14$'$}$$

The elimination of t between the equations (21)$'$ and (14)$'$ gives the elliptic cone

$$\frac{1}{\theta^2} = \frac{\cos^2\phi_1}{\theta_0^2} + \frac{\sin^2\phi_1}{\varepsilon^2}. \tag{26}$$

The intersection of this cone with a horizontal plane at the distance $\sqrt{A}$ from the point O and fixed with regard to the cone (26), gives an ellipse with the semiaxes

$$a = \sqrt{A}\,\theta_0,$$
$$b = \sqrt{A}\,\varepsilon = \frac{2A - C}{2A}\sqrt{\frac{A}{Mgl}}\,a\omega \sin\lambda.$$

A first approximation gives for N the value $-\frac{2A-C}{2A}\sin\lambda$. These results are in perfect accordance with those of Mr. Kamerlingh Onnes.[*]

A more accurate value of N is obtained by retaining second powers of θ_0. Formula (22) then gives

$$N = -\frac{2A-C}{2A}\sin\lambda + \frac{C}{2A}\theta_0\cos\lambda\sin\phi_0 + \frac{3}{8}\frac{A-C}{A}\theta_0^2\sin\lambda,$$

which differs from the result obtained by Mr. Kamerlingh Onnes, who gives the value

$$N = -\frac{2A-C}{2A}\sin\lambda + \frac{C}{2A}\theta_0\cos\lambda\sin\phi_0 + \frac{3}{8}\frac{2A-C}{2A}\theta_0^2\sin\lambda.$$

[†] Over de betrekkelijke Beweging. Nieuw Archief voor Wiskunde, Deel V, p. 162.

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A Method for Calculating Simultaneously all the Roots of an Equation.[*]

By Emory McClintock.

The comprehensive method which forms the subject of this paper may be introduced best by practical illustrations, beginning with trinomials and proceeding later to equations in general. Let it be desired to learn approximately all the roots of the equation $x^6 = -1 - x$. In this case we may use the formula

$$x = \omega - \omega^2 a - \tfrac{3}{2}\omega^3 a^2 - \tfrac{8}{3}\omega^4 a^3 - \ldots\ldots, \tag{1}$$

where $a = -\frac{1}{6}$, and ω is any one of the sixth-roots of -1, viz.

m	ω_1^m	ω_2^m	ω_3^m	ω_4^m	ω_5^m	ω_6^m
1	$\sqrt{-1}$	$-\sqrt{-1}$	$\frac{1}{2}\sqrt{3}+\frac{1}{2}\sqrt{-1}$	$\frac{1}{2}\sqrt{3}-\frac{1}{2}\sqrt{-1}$	$-\frac{1}{2}\sqrt{3}+\frac{1}{2}\sqrt{-1}$	$-\frac{1}{2}\sqrt{3}-\frac{1}{2}\sqrt{-1}$
2	-1	-1	$\frac{1}{2}+\frac{1}{2}\sqrt{-3}$	$\frac{1}{2}-\frac{1}{2}\sqrt{-3}$	$\frac{1}{2}-\frac{1}{2}\sqrt{-3}$	$\frac{1}{2}+\frac{1}{2}\sqrt{-3}$
3	$-\sqrt{-1}$	$\sqrt{-1}$	$\sqrt{-1}$	$-\sqrt{-1}$	$\sqrt{-1}$	$-\sqrt{-1}$
4	1	1	$-\frac{1}{2}+\frac{1}{2}\sqrt{-3}$	$-\frac{1}{2}-\frac{1}{2}\sqrt{-3}$	$-\frac{1}{2}-\frac{1}{2}\sqrt{-3}$	$-\frac{1}{2}+\frac{1}{2}\sqrt{-3}$
5	$\sqrt{-1}$	$-\sqrt{-1}$	$-\frac{1}{2}\sqrt{3}+\frac{1}{2}\sqrt{-1}$	$-\frac{1}{2}\sqrt{3}-\frac{1}{2}\sqrt{-1}$	$\frac{1}{2}\sqrt{3}+\frac{1}{2}\sqrt{-1}$	$\frac{1}{2}\sqrt{3}-\frac{1}{2}\sqrt{-1}$
6	-1	-1	-1	-1	-1	-1

By inserting in (1) the numerical value of a we have

$$x = \omega + \tfrac{1}{6}\omega^2 - \tfrac{1}{24}\omega^3 + \tfrac{1}{81}\omega^4 - \ldots\ldots$$

For the several pairs of roots, or values of x, we have therefore, taking the values of ω as stated, and remembering that $\sqrt{3} = 1.732$ nearly,

$$x = -\tfrac{1}{6} \quad + \tfrac{1}{81} - \quad \ldots\ldots \pm \sqrt{-1}\,(1 + \tfrac{1}{24} \ldots\ldots) = -\tfrac{25}{162} \pm \tfrac{25}{24}\sqrt{-1}, \text{ nearly};$$

$$x = \quad \tfrac{1}{2}\sqrt{3} + \tfrac{1}{12} - \tfrac{1}{162} \ldots\ldots \pm \sqrt{-1}\,(\tfrac{1}{2} + \tfrac{1}{12}\sqrt{3} - \tfrac{1}{24} + \tfrac{1}{162}\sqrt{3} \ldots\ldots)$$
$$= .94 \pm .61\sqrt{-1}, \text{ nearly};$$

$$x = -\tfrac{1}{2}\sqrt{3} + \tfrac{1}{12} - \tfrac{1}{162} \ldots\ldots \pm \sqrt{-1}\,(\tfrac{1}{2} - \tfrac{1}{12}\sqrt{3} - \tfrac{1}{24} - \tfrac{1}{162}\sqrt{3} \ldots\ldots)$$
$$= -.79 \pm .30\sqrt{-1}, \text{ nearly}.\dagger$$

---

[*] Read before the American Mathematical Society on August 14 and October 27, 1894. The portion read on August 14 is indicated in the Society's Bulletin for October, 1894.

[†] This example has been employed by Spitzer (Allgemeine Auflösung der Zahlengleichungen, Wien, 1851) and by Jelinek (Die Auflösung der höheren numerischen Gleichungen, Leipzig, 1865), to illustrate

The formula (1) made use of is valid only for equations of the form $x^6 = -1 + 6ax$, where a is numerically less than the sixth-root of 5^{-5}. The same formula may be used, with the same restriction upon the value of a, for equations of the form $x^6 = 1 - 6ax$, by taking ω for any sixth-root of 1. It is a special case of a more general formula, applicable to all equations of the form $x^n = \omega^n + nax^{n-k}$ (that is to say, all trinomial equations) for which the series is convergent:

$$\begin{aligned} x = \omega + \omega^{1-k}a &+ \omega^{1-2k}(1-2k+n)\,a^2/2\,! \\ &+ \omega^{1-3k}(1-3k+n)(1-3k+2n)\,a^3/3\,! \\ &+ \omega^{1-4k}(1-4k+n)(1-4k+2n)(1-4k+3n)\,a^4/4\,! + \dots\dots \end{aligned} \qquad (2)$$

Here ω may have any value, but a^n must for convergency be smaller numerically than $k^{-k}(n-k)^{k-n}\omega^{nk}$ when n is positive. If n is negative, a^{-n} must for convergency be smaller numerically than $(-k)^k(k-n)^{n-k}\omega^{-nk}$. Nothing is gained by having n negative, since in that event we have only to multiply $x^n = \omega^n + nax^{n-k}$ by $x^{-n}\omega^{-n}$ to produce the positive form $x^{-n} = \omega^{-n} - nbx^{-k}$, where $b = a\omega^{-n}$.

In the interpretation of the trinomial formula (2), it is usually most convenient to reduce the given trinomial $x^n = \omega^n + nax^{n-k}$ to that form in which ω^n is 1 or -1, in which case ω means merely any n^{th} root of 1 or -1, as the case may be. If that is not done, we must interpret ω as equivalent to $c\xi$, where c is the n^{th} root of the numerical value of ω^n, and ξ is any n^{th} root of 1 or -1, according as ω^n is positive or negative. This remark will apply to other equations as well as to trinomials. For uniformity of illustration, the examples adduced will be of the form $\omega^n = 1$ or $\omega^n = -1$, to which form any equation $x^n = \pm c^n + f(x)$ is at once reduced by writing cx for x.

Since its first use by Newton, if Newton was its author, no discussion of numerical equations can be considered complete without introducing the celebrated equation $x^3 - 2x - 5 = 0$.[*] To this as it stands we can apply the trinomial formula (2) at once, but a more convergent series may be had by suppressing the term next to the last, by the transformation $x^{-1} = y - 2/15$, so that the equation to be solved becomes $3375\,y^3 - 180\,y - 659 = 0$. In this

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the changes made by them in Horner's method to make it serve in the computation of imaginary roots. Such a calculation as theirs can only be done for one pair at a time, and that with considerable difficulty, after first assigning approximate locations.

[*] Employed after Newton for successive new methods by Lagrange, Fourier, Sturm, and Murphy, not to speak of other writers.

let $y = (659/3375)^{\frac{1}{3}} z = .580146\, z$,* so that $z^3 = 1 + 3az$, where $a = .0528206$. Here $\omega^3 = 1$, and by (2) we have

$$z = \omega + \omega^2 a - \tfrac{1}{3}\omega a^3 + \dots .$$
$$= \omega + .05282\omega^2 - .00005\omega + \dots . = .99995\omega + .05282\omega^2 \dots .$$

The cube roots of 1 being 1 and $-\frac{1}{2} \pm \frac{1}{2}\sqrt{-3}$, the three values of z are therefore 1.05277 and $-.52639 \pm .47356\sqrt{-3}$, whence those of y are .60176 and $-.30088 \pm .47585\sqrt{-1}$, all correct to the last place. The real root only has heretofore been sought by the various writers who have dealt with this equation, so far as appears, except Murphy, who finds approximate values of the imaginary roots by two methods, both tedious.†

Murphy adds another example, occupying two or three pages, concerning which, writing in 1838, he says: ". . . . as this method may be said to be the only direct one known for obtaining a first notion of the magnitudes of the real and imaginary parts of the roots of equations, we have, therefore, developed it at length . . . .; we may add that the research of the impossible roots of equations has been generally overlooked in modern treatises of algebra."‡ The equation which he employs is $x^4 + x + 10 = 0$, and he finds eventually, for the four roots, $1.251 \pm 1.348\sqrt{-1}$ and $-1.251 \pm 1.282\sqrt{-1}$. If we take $x = 10^{\frac{1}{4}} y = 1.77828y$, the equation to be solved becomes $y^4 + 10^{-\frac{3}{4}} y + 1 = 0$, or $y^4 = -1 + 4ay$, where $a = -.044457$. Applying (2), and observing that $\omega^4 = -1$, $n = 4$, $k = 3$,

$$y = \omega + \omega^{-2}a - \tfrac{1}{2}\omega^{-5}a^2 + \dots . = \omega - \omega^2 a - \tfrac{1}{2}\omega^3 a^2 + \dots .$$
$$= \omega + .044457\omega^2 - .000988\omega^3 + \dots .$$

The first term neglected is $-\frac{7}{8}\omega a^4$, so that the terms taken should be good to five places. When $\omega^4 = -1$, we have either $\omega = \sqrt{\frac{1}{2}}(1 \pm \sqrt{-1})$, $\omega^2 = \pm\sqrt{-1}$, $\omega^3 = \sqrt{\frac{1}{2}}(-1 \pm \sqrt{-1})$; or, $\omega = \sqrt{\frac{1}{2}}(-1 \pm \sqrt{-1})$, $\omega^2 = \mp\sqrt{-1}$, $\omega^3 = \sqrt{\frac{1}{2}}(1 \pm \sqrt{-1})$. Also, $\sqrt{\frac{1}{2}} = .707107$. Taking the first pair of values of ω, we have

$$y = .707107\,(1 + .000988) \pm .044457\sqrt{-1} \pm .707107\,(1 - .000988)\sqrt{-1}$$
$$= .70781 \pm .75086\sqrt{-1},$$

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* **Decimal fractions which are only approximately true will be stated, for convenience, without any such qualifying phrase as "nearly."**

† ***Theory of Equations*, p. 124; pp. 185–188.**

‡ **Todhunter remarks that "there is no easy practical method of calculating the imaginary roots of equations at present known." Cayley, 1878, in the article "Equation" in the Encyclopædia Britannica, says: "Very little has been done in regard to the calculation of the imaginary roots of an equation by approximation; and the question is not here considered."**

and similarly for the second pair,

$$y = -.70781 \pm .66195\sqrt{-1}.$$

The four values of $x = 1.77828y$ are therefore $x = 1.2587 \pm 1.3352\sqrt{-1}$ and $x = -1.2587 \pm 1.1771\sqrt{-1}$. These values are very closely correct, so that Murphy may possibly have made some error, though he says distinctly that his results must be taken as only a first approximation.

Enough has been said to illustrate the use of (2) when the given trinomial is fit—a phrase which will be used to express readiness for the application of the methods now brought forward so as to yield a convergent series. Murphy's quartic was found fit at once, as was the sextic first discussed. Newton's cubic was improved by a linear transformation; a process not usually available, when a trinomial is desired, for degrees above the third. Two questions therefore arise: what shall be done to improve a given cubic, and what shall be done when the degree is higher and the trinomial is unfit?

If the cubic $x^3 = \pm 1 + 3ax$ has $a^3 < \frac{1}{4}$,* it is fit, though a transformation may secure greater convergency. If $a^3 = \frac{1}{4}$, a being positive, there are two equal roots, and no transformation will avail. If $a^3 \geqq \frac{1}{4}$, and if a is negative, the transformation $x^{-1} = y \mp a$ will serve, though some other may serve better. If $a^3 > \frac{1}{4}$, and if a is positive, the roots of the cubic are all real, and the exhibition of all of them at once is impossible, since the formula (2) expressly contemplates, for cubics, two imaginary roots, corresponding to the two imaginary cube roots of unity employed. In this latter case the trinomial is radically unfit, and the disposition to be made of it may be considered along with that of unfit trinomials of higher degrees, as part of the second question.

Except for improvable cubics, as just explained, there is usually nothing better to be done with an unfit trinomial, as a trinomial, than to apply the trinomial formula (2) twice, securing $n - k$ roots by one operation, and the remaining k roots by the second; though sometimes the trinomial form may be abandoned advantageously. The original trinomial being $x^n = \omega^n + nax^{n-k}$, we now regard it, for the first operation, as $x^{n-k} = \omega_1^{n-k} + (n-k)a_1x^n$, where $\omega_1^{n-k} = -\omega^n/na$, and a_1 is the reciprocal of $(n-k)na$. More simply, if $x^n = \pm 1 + nax^{n-k}$, $x^{n-k} = (\mp 1 + x^n)/na$. That $n - k$ roots can now be produced by a convergent series is readily proved. By the supposition of unfitness,

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* Numerical values always understood.

$a^n > k^{-k}(n-k)^{k-n}\omega^{nk}$, and we have also $a = n^{-1}(n-k)^{-1}a_1^{-1}$, $\omega^n = -na\omega_1^{n-k} = a_1^{-1}\omega_1^{n-k}/(k-n)$. Hence $n^{-n}(n-k)^{-n}a_1^{-n} > k^{-k}(n-k)^{k-n}a_1^{-k}\omega_1^{k(n-k)}(k-n)^{-k}$, whence $a_1^{n-k} < n^{-n}(-k)^k\omega_1^{-k(n-k)}$, which satisfies the criterion of convergency first stated, it being observed that $n-k$ now takes the place of n, n that of $n-k$, and $-k$ that of k.

For example, let

$$4y^3 - 243y + 165 = 0. \tag{3}$$

Here $y^3 = -165/4 + 3(81/4)y$, and the numerical value of $(81/4)^3$ is not less than that of $2^{-2}1^{-1}(165/4)^2 = (165/8)^2$. We must therefore, since a is positive, employ two operations: one upon the equation $y = 165/243 + 4y^3/243$, and the second upon the equation $y^2 = 243/4 - 165y^{-1}/4$, the first yielding one real root, the second yielding the two remaining real roots. For the first, let $y = 55u/81$, so that the equation reduces to $u = 1 + 4.55^2.3^{-1}.81^{-3}u^3 = 1 + .00759u^3$. Here $a = .00759$, $n = 1$, $k = -2$, and from (2) we have

$$u = 1 + a + 3a^2 + \ldots. = 1 + .00759 + .00017 = 1.00776.$$

Hence $y = .684$. For the second, let $y = \frac{1}{2}243^{\frac{1}{2}}v$, so that the equation reduces to $v^2 = 1 - 330(243)^{-\frac{3}{2}}v^{-1} = 1 - .08712v^{-1}$, or $v^2 = 1 + 2av^{-1}$, where $a = -.04356$. By (2), taking $n = 2$, $k = 3$,

$$v = \omega - \omega^{-2}.04356 - \omega^{-5}.00285 \ldots. = .997\omega - .044.$$

As ω may be 1 or -1, v is .953 or -1.041; and as $y = 7.794v$, the remaining values of y are 7.43 and -8.11.

In this example we have illustrated not only the first operation, securing $n-k\,(=1)$ root, but also the second operation, securing the remaining $k\,(=2)$ roots. For the latter the formula used is equivalent to $x^k = na + \omega^n x^{k-n}$, obtained by multiplying both sides of the original trinomial $x^n = \omega^n + nax^{n-k}$ by x^{k-n}. For the simpler form $x^n = \pm 1 + nax^{n-k}$, we have $x^k = na \pm x^{k-n}$. That this transformation necessarily yields a convergent series when the original trinomial is unfit may be seen upon due substitution. For, putting $\omega^n = ka_1$ and $a = n^{-1}\omega_1^k$, we have, as the criterion of convergency of the derived equation $x^k = \omega_1^k + ka_1x^{k-n}$, the requirement $a_1^k < n^{-n}(k-n)^{n-k}\omega_1^{nk}$, and this is satisfied by substituting for ω^n and a their values in the known numerical inequality $a^n > k^{-k}(n-k)^{k-n}\omega^{nk}$.

Referring to the last example (3), we took first $n=1$. Whenever $n=1$ the trinomial equation $x^n=\omega^n+nax^{n-k}$ becomes $x=\omega+ax^m$, putting $k=1-m$. Then (2) becomes, for this special case,

$$x=\omega+\omega^m a+\omega^{2m-1}ma^2+\omega^{3m-2}m\,(m-1)\,a^3/2\,!+\ldots.\,, \tag{4}$$

a series known to Euler and Lagrange, and immediately derivable from Lagrange's theorem,

$$f(x)=f(\omega)+\phi\,(\omega)f'\,(\omega).\,a+\frac{d}{d\omega}\,\{[\phi\,(\omega)]^2f'\,(\omega)\}.\,a^2/2\,!+\ldots.\,, \tag{4*}$$

where the relation is $x=\omega+a\phi\,(x)$. The trinomial series (2) may be derived at once, as will be seen, from Lagrange's theorem (4*), by putting $x^n=u$, so that the relation is $u=\omega^n+nau^{1-k/n}$, and expanding $f\,(u)=u^{1/n}$ by means of (4*). This is however an afterthought, the series (2) having been discovered, without employing Lagrange's theorem, in the course of writing the preceding paper ("Theorems in the Calculus of Enlargement") together with and as a case of a more general series which applies to other as well as to trinomial equations.

The new theorems numbered (21) and (22) in the paper just referred to are, when $x^n=\omega^n+na\phi\,(x)$,

$$\begin{aligned}x=\omega+\omega^{1-n}\phi\omega.\,a+\mathrm{E}_0 0\,(\phi\mathrm{E}_0)^2\,\omega^{0-2n}\,(0-n).\,a^2/2\,!&\\ +\,\mathrm{E}_0 0\,(\phi\mathrm{E}_0)^3\,\omega^{0-3n}\,(0-n)(0-2n).\,a^3/3\,!+\ldots.\,,&\end{aligned} \tag{5}$$

$$\begin{aligned}x=\omega+\omega^{1-n}\phi\omega.\,a+\omega^{1-n}\frac{d}{d\omega}\,\omega^{1-n}\,(\phi\omega)^2.\,a^2/2\,!&\\ +\left(\omega^{1-n}\frac{d}{d\omega}\right)^2\omega^{1-n}\,(\phi\omega)^3.\,a^3/3\,!+\ldots.&\end{aligned} \tag{6}$$

Here the brackets about the letter following the functional sign ϕ are omitted for convenience. The two theorems (5) and (6) are of course identical, except as to the form of the coefficients, which have the same real value, each to each. The symbol 0 is equivalent to t, when $t=0$, and the symbol of operation E_0 is to be interpreted as that by which, when applied to any function of 0, say $f0$, the latter becomes the same function of $0+1$, say $f(0+1)$. Similarly, this operation being subject to the law of indices, $\mathrm{E}_0^h f0=f(0+h)$; and more generally, if $\phi\mathrm{E}_0=a\mathrm{E}_0^n+b\mathrm{E}_0^{n-1}+\ldots.\,,\ \phi\mathrm{E}_0f0=a\mathrm{E}_0^nf0+b\mathrm{E}_0^{n-1}f0+\ldots.=af\,(0+n)+bf(0+n-1)+\ldots.$ So also with powers of $\phi\mathrm{E}_0$; if for instance $\phi\mathrm{E}_0=\mathrm{E}_0^n+\mathrm{E}_0^m$, $(\phi\mathrm{E}_0)^2f0=(\mathrm{E}_0^{2n}+2\mathrm{E}_0^{n+m}+\mathrm{E}_0^{2m})\,f0=f\,(0+2n)+2f\,(0+n+m)+f\,(0+2m)$.

If, for example, in (5), $\phi x=x^{n-k}$, so that $\phi\mathrm{E}_0=\mathrm{E}_0^{n-k}$, the coefficient of a is $\omega^{1-n}\omega^{n-k}=\omega^{1-k}$, as in (2). Again, the coefficient of $a^2/2\,!$ is $\mathrm{E}_0 0\mathrm{E}_0^{2n-2k}\omega^{0-2n}\,(0-n)$

$= \mathrm{E}_0 0\omega^{0-2k}(0+n-2k) = (0+1)\omega^{0+1-2k}(0+1+n-2k) = \omega^{1-2k}(1-2k+n)$, as in (2); and the other coefficients of (2) will be confirmed in like manner, so that (2) is a special case of (5). It is likewise a special case of the conjugate expression (6), which is derived from Lagrange's theorem (4*) by writing x^n, ω^n, and na, for x, ω, and a respectively, and taking $x = f(x^n) = (x^n)^{\frac{1}{n}}$. This use of Lagrange's theorem has escaped notice since 1768, when that theorem was published, and might have escaped notice much longer had it not been for the circumstance that the relation underlying (5), from the nature of its origin, is $x^n = \omega^n + na\phi x$ instead of $x = \omega + a\phi x$. Given this relation and this series (5), it is most natural to remark that ω must have n values, determined by the nth roots of 1 or of -1.

The theorem (5) is itself a special case of the more general theorem, numbered (19) in the preceding paper,

$$fx = f\omega + f\mathrm{E}_0 0\phi\mathrm{E}_0\omega^{0-n}.a + f\mathrm{E}_0 0(\phi\mathrm{E}_0)^2\omega^{0-2n}(0-n).a^2/2! + \ldots., \tag{7}$$

from which

$$x^m = \omega^m + m\omega^{m-n}\phi\omega.a^{*} + m(\phi\mathrm{E}_0)^2\omega^{0+m-2n}(0+m-n).a^2/2! + \ldots., \tag{8}$$

the given relation being still $x^n = \omega^n + na\phi x$. By means of this equation we can find approximations, in convergent cases, to the mth powers of the several roots of a given equation $x^n = \omega^n + na\phi x$, by employment of the nth roots of 1 or of -1. Apart from the case $m = 1$, however, I see no practical use for any other case than $m = -1$. In this case (8) becomes

$$x^{-1} = \omega^{-1} - \omega^{-1-n}\phi\omega.a - (\phi\mathrm{E}_0)^2\omega^{0-1-2n}(0-1-n).a^2/2! - \ldots. \tag{9}$$

For the trinomial case $x^n = \omega^n + nax^{n-k}$, this becomes

$$x^{-1} = \omega^{-1} - \omega^{-1-k}.a - \omega^{-1-2k}(-1-2k+n).a^2/2! - \ldots. \tag{10}$$

Before proceeding to discuss the principal series (5), it will be well to consider finally the reciprocal series (9) and (10). It is obvious that in any of the trinomial examples already brought forward we might, with little additional difficulty, by the use of (10), obtain different approximations to the roots by means of their reciprocals. Take, for instance, the equation numbered (3), where $u = 1 + .00759u^3$. Here $n = 1$, $k = 2$, $a = .00759$, and from (10)

$$u^{-1} = 1 - a - 2a^2 \ldots. = 1 - .00759 - .00012 = .99229.$$

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* Here $m\phi\mathrm{E}_0\omega^{0+m-n} = m\omega^{m-n}\phi\mathrm{E}_0\omega^0$, and $\phi\mathrm{E}_0\omega^0 = \phi\omega$, by a known theorem.

This differs but 1 in the last place from the reciprocal of the value found for u, 1.00776, a matter of no importance considering that the result is desired to three places only. Cases may arise in which these reciprocal formulæ will be found valuable; and they will certainly be found of the highest value if and when means are discovered for distinguishing those cases in which the reciprocal approximation is the more correct. Little appears to be gained, on the average, by taking a mean between the values ascertained by the direct and reciprocal approximations; yet in half of all cases the reciprocal approximation must be the closer of the two. Probably in most cases both the direct and the reciprocal approximations will err on the same side of the true value; and as regards the contrary chance, it seems better to continue one approximation further than to go to the labor of computing both. At present, therefore, no practical advantage appears to be derived from the use of these reciprocal theorems, though further investigation may enable them to take the place, in suitable cases, of the direct. It will be found, on examination, and may readily be proved, that the same series is obtained by (5) or (6) for x, from $x^n = \omega^n + na\phi x$, as by (9) for $x = y^{-1}$ from $y^n = \omega^{-n} - na\omega^{-n} y^n \phi(y^{-1})$. The differential formulæ which correspond to (7), (8), and (9) as (6) corresponds to (5) are, respectively,

$$fx = f\omega + \omega^{1-n}\phi\omega f'\omega . a + \left(\omega^{1-n}\frac{d}{d\omega}\right)\left[\omega^{1-n}(\phi\omega)^2 f'\omega\right] . a^2/2! + \ldots. , \qquad (11)$$

$$x^m = \omega^m + m\omega^{m-n}\phi\omega . a + m\left(\omega^{1-n}\frac{d}{d\omega}\right)\left[\omega^{m-n}(\phi\omega)^2\right] . a^2/2! + \ldots. , \qquad (12)$$

$$x^{-1} = \omega^{-1} - \omega^{-1-n}\phi\omega . a - \left(\omega^{1-n}\frac{d}{d\omega}\right)\left[\omega^{-1-n}(\phi\omega)^2\right] . a^2/2! - \ldots. \qquad (13)$$

Returning to the general solution (5), for which hereafter the reader may at his option substitute (6) as equivalent, we have seen that when $\phi x = x^{n-k}$, that is, when the given equation $x^n = \omega^n + na\phi x$ is a trinomial, the general solution takes the form of (2). For all cases in which (2) is not available, and in fact for all cases whether (2) is available or not, the solution is equally general if $a = 1$. Again, since $E_0 0 = 1$, the first zero in each term of (5) may be omitted. We shall therefore, now that we are passing beyond the consideration of trinomials as such, write the general solution of $x^n = \omega^n + n\phi x$ thus:

$$x = \omega + \omega^{1-n}\phi\omega + \tfrac{1}{2}(\phi E_0)^2 \omega^{0+1-2n}(0+1-n)$$
$$+ \tfrac{1}{6}(\phi E_0)^3 \omega^{0+1-3n}(0+1-n)(0+1-2n) + \ldots. \qquad (14)$$

Knowing this formula, we are usually enabled, almost at a glance, to determine the number of imaginary roots, the signs of the real roots, and often the signs of the real parts of the imaginary roots, of any given equation; and applying it, we can obtain with little difficulty approximations more or less exact to the values of all the roots. This means that Sturm's theorem, acknowledged hitherto to be the only complete solution of the problem of separation of roots, is no longer essential in the examination of numerical equations. To make this clear, and also to avoid the charge, brought by Fourier against Euler, of selecting easy examples, I shall shortly take up in order all of the illustrations, eleven in number, employed in Burnside and Panton's *Theory of Equations* in the section devoted to the application of Sturm's theorem.

Having an equation presented for solution, the first step is to inspect it, to see whether it is fit. If not, some linear transformation must be sought to render it fit. If that prove impracticable, the inventor of the equation may be suspected of introducing equal roots and known tests may be applied. (With coefficients taken at random, equal roots are not likely to appear.) If no equal roots are found, a suitable transformation is possible. No equation is certainly fit, however, unless all the roots can be found by convergent series.

I shall use the word "span" for the degree of an operation, represented by the letter n in $x^n = \omega^n + n\phi x$. Thus, for Jelinek's equation, $x^6 + x + 1 = 0$, the span is sextic, and the equation is solved with one span. For Newton's equation, $x^3 - 2x - 5 = 0$, there is but a single span, a cubic; and for Murphy's equation, $x^4 + x + 10 = 0$, likewise but one, a quartic. For the equation numbered (3), however, $4y^3 - 243y + 165 = 0$, two spans are needed, namely, from left to right, a quadratic and a linear span. Examining these trinomials, we see that in the first three the middle coefficient is small enough to permit a single span, while in the last it is too large. I say therefore that in the first three cases the "dominant" coefficients are the first and last, while in the last case all three coefficients are "dominant." A span stretches from one dominant to the next. There is therefore but one span for each of the first three equations, while two are indicated for the last. If the dominants which define any span have like signs, it is a "like span"; otherwise an "unlike span." The recognition of dominants is not always easy, but is often facilitated by some simple transformation. Once recognized, they disclose the nature of all the roots at a glance. If exceptions exist to this statement, they have yet to be discovered. Each span represents n roots. It will be remembered that ω^n is positive for a span whose dominants have unlike signs, and *vice versa*. The following table, showing the number of different kinds of roots covered by each span, is based on the characteristics of the n^{th} roots of 1 or of -1:

Description of Span.	Real Roots.		Imaginary Roots.		
	Positive.	Negative.	Positive.	Negative.	Uncertain.
$n = 1$, unlike signs:	1				
like:		1			
$n = 2$, unlike:	1	1			
like:					2
$n = 3$, unlike:	1			2	
like:		1	2		
$n = 4$, unlike:	1	1			2
like:			2	2	
$n = 5$, unlike:	1			2	2
like:		1	2		2
$n = 6$, unlike:	1	1	2	2	
like:			2	2	2

Recurring to the examples given, we find for $x^6 + x + 1 = 0$, $n = 6$, signs like, therefore six imaginary roots, two of them at least with real parts positive, two at least negative; for $x^3 - 2x - 5 = 0$, $n = 3$, signs unlike, one real positive, two imaginary with real parts negative; for $x^4 + x + 10 = 0$, $n = 4$, like signs, two pairs of imaginary roots, the real parts having opposite signs; and for that numbered (3), $4y^3 - 243y + 165 = 0$, first span, $n = 2$, signs unlike, therefore two real roots of opposite signs, to which the second span, $n = 1$, again unlike, adds another real positive.

A recognition of the dominants of any equation not only indicates the nature of the roots, but shows how, if desired, we can proceed first to the computation of the greatest or least root. It will be found that the first span of n_1 terms yields not only n_1 roots, but the n_1 greatest of the roots; that the next span of n_2 terms yields the n_2 roots next greater in size, and so on, till the last span of n_k terms, which must yield the n_k roots of smallest size. In this statement imaginary roots rank according to their moduli. For, let a, b, c be the numerical values of successive dominants, always counting from left to right, with the highest power of the unknown quantity, as usual, on the left. The point to be shown is that the roots covered by the span from a to b, of k degrees, are all larger than the largest of those covered by the span from b to c, of l degrees.

That b shall be a dominant requires that b^{k+l} shall be decidedly larger than $a^l c^k$; let this be assumed. To put these spans successively in the form $v^n = \pm 1 + n\phi v$, we first put $n = k$, $x = (b/a)^{1/k} v_1$, and afterwards $n = l$, $x = (c/b)^{1/l} v_2$. As regards the first span, the series is convergent for the k values of v_1 which are nearest to unity; and as regards the second span, the series is convergent for the l values of v_2 which are nearest to unity. Since $(b/a)^{1/k}$ is decidedly larger than $(c/b)^{1/l}$, the values of x corresponding to the k values of v_1 must be larger than the largest of those which correspond to the l values of v_2. That any one root is not presented in each of two adjacent spans, is immediately assented to when we reflect that if it were, some other root must fail of presentation at all. That this intuitive assent is correct may readily be seen. Let us first suppose the n-equation to be a trinomial of one span. Let a root or pair of roots change continuously so that the span becomes less and less convergent, and finally not convergent but divergent: a new dominant has arisen, dividing the roots into two classes, k and $n - k$ in number respectively. Let now the latter span be again subdivided without destroying the dominant in question: the k roots of the other series remain together. As another supposition, let us for the sake of argument, imagine one and the same root to be in two adjacent spans, on both sides of a given dominant. Let all the other middle dominants gradually subside by changes of other roots until we have a new trinomial with the given root still on both sides of the middle dominant: the situation is absurd, and it is equally absurd to suppose any root to be either annihilated or created by gradual continuous change of other roots.

It is to be observed that coefficients are not to be counted as "dominant," in the sense here employed, unless they are relatively of larger size. It is possible that of three coefficients, a, b, c, we may find b relatively small, and that we may also find a convergent series for the span from b to c: if so, the order of magnitude will be disarranged. Cases of that sort are seldom likely to be observed, however, unless specially invented. Lagrange laid it down that the single root of the equation $x = \omega + \phi x$ developed by his method of 1768 (a method equivalent to that special case of the present general method in which $n = 1$) is always the smallest root; but his statement is not always correct unless the coefficient of x is a dominant, in the sense here used.*

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* That Lagrange's proof concerning the smallest root is incomplete was shown by Chio, in a memoir presented to the Institute of France in 1846. Chio had however apparently no idea of "dominants." That the smallest root can easily be found when the coefficient of x is relatively very large is of course familiar.

Each of the paragraphs which follow, having attached to them numbers, from 1 to 11, relates to the example having the same number in the work on equations already mentioned.

[1]. $x^4 + 3x^3 + 7x^2 + 10x + 1 = 0$. The dominants here are uncertain. It cannot be claimed, as obvious, either that they are 1, 10, 1; 1, 7, 10, 1; or 1, 3, 10, 1. In these cases respectively the spans would be either a like cubic and a like linear; a like quadratic and two like linears; or a linear, a quadratic, and a linear, all with like signs. But any one of these combinations would indicate two real negative roots and two imaginary, one of the real roots being the smallest of the four. When the dominants are at all doubtful the equation is unfit and must be transformed. Let $x = y - 1$; then $y^4 - y^3 + 4y^2 + y - 4 = 0$. Spans, two quadratics, respectively like and unlike; whence y has two (larger) imaginary values and two (smaller) real values of opposite signs. From this equation in y we find approximately that the four values of x are, $-.4 \pm 2.2\sqrt{-1}$ and $-1 \pm .9$, showing that the largest values are imaginary, which rules out 1, 3, 10, 1 as the dominants, and that the real parts are negative, which rules out 1, 10, 1; leaving 1, 7, 10, 1 as correct. For the details, beginning with the first y-span, let $y = 2u$; then $u^2 = -1 + \frac{1}{2}u - \frac{1}{8}u^{-1} + \frac{1}{4}u^{-2} = -1 + 2\phi u$, where $\omega^2 = -1$, $\phi u = (4u - u^{-1} + 2u^{-2})/16$, and by (14)

$$\begin{aligned} u &= \omega + \omega^{-1}\phi\omega + \dots\dots = \omega + \omega^{-1}(4\omega - \omega^{-1} + 2\omega^{-2})/16 \\ &= 5/16 + 18\omega/16. \end{aligned}$$

Hence, for a rough approximation, $u = .3 \pm 1.1\sqrt{-1}$, so that $y = .6 \pm 2.2\sqrt{-1}$ or thereabouts, and $x = -.4 \pm 2.2\sqrt{-1}$. To approximate to the real values, by operating upon the second span, we have $y^2 = 1 + \frac{1}{4}(y^3 - y - y^4) = 1 + 2\phi y$, where $\phi y = \frac{1}{8}(y^3 - y - y^4)$. Hence $y = \omega + \omega^{-1}\phi\omega + \dots\dots$, where $\omega = \pm 1$, or $y = \omega + \frac{1}{8}(\omega^2 - 1 - \omega^3) + \dots\dots$, whence $y = \pm .9$, and $x = \pm .9 - 1$. A better transformation, if close results were desired, could be got by putting $y^{-1} = v + 1/16$.

[2]. $x^4 - 4x^3 - 3x + 23 = 0$. Dominants, 1, -4, 23; indicating a linear span, unlike, and a cubic, also unlike; largest root real positive, another real positive, two imaginary. For a fitter equation, take $x = 2(1 + u)/(1 - u)$, so that $77u^4 + 24u^3 + 234u^2 - 80u + 1 = 0$. Here the spans are, quadratic with like signs, and two linear, each unlike; hence two larger roots imaginary, two smaller real and positive. If $u = (234/77)^{\frac{1}{2}}y = 1.74y$, we shall have

$y^2 + .18y + 1 - .20y^{-1} = 0$, neglecting the term in y^{-2}, whence $y^2 = -1 + 2(-.09y + .10y^{-1})$. Here $\omega = \pm\sqrt{-1}$, and $y = -.19 + \omega = -.19 \pm \sqrt{-1}$, so that $u = -.33 \pm 1.75\sqrt{-1}$ or thereabouts. For the larger real root, let $u = 40z/117$, whence $z = 1 - .04z^3 - .03z^2 - .04z^{-1}$. In the second member we may replace z by $\omega = 1$, whence $z = .89$, $u = .31$. For the smallest root, let $u = v/80$, and we find similarly $v = 1.04$, $u = .013$.

[3]. $2x^4 - 13x^2 + 10x - 19 = 0$. Dominants, 2, -13, -19: two quadratic spans, unlike and like respectively; hence two larger roots real with opposite signs, two smaller imaginary. Take $x^{-1} = 1 - y$, and again $y^{-1} = .1v + .9$, whence $x = (v + 9)/(v - 1)$. Then $v^4 - v^2 - 402v - 598 = 0$, which is to be solved. Here the spans are an unlike cubic and a like linear. The smallest root is real negative, the others are one real positive, two imaginary. For the cubic span, let $v = (402)^{\frac{1}{3}} z = 7.38z$, so that $z^4 - .018z^2 - z - .198 = 0$, and $z^3 = 1 + 3(.006z + .066z^{-1})$. The values of ω are 1 and $-\frac{1}{2} \pm .866\sqrt{-1}$, and the corresponding values of ω^2 are 1 and $-\frac{1}{2} \mp .866\sqrt{-1}$. We have here $z = \omega + \omega^{-2}(.006\omega + .066\omega^{-1}) = .066 + \omega + .006\omega^2$, and the values of z are 1.07 and $-.44 \pm .86\sqrt{-1}$, whence those of v are 7.9 and $-3.2 \pm 6.3\sqrt{-1}$. For the smallest root, let $v = 299u/201$, whence by the usual linear process $u = -1 - 3/640$ nearly, $v = -1.5$.

[4]. $x^5 + 2x^4 + x^3 - 4x^2 - 3x - 5 = 0$. A quintic equation with a full quintic span, two terminal dominants only, with unlike signs: one real positive root, four imaginary. To secure a fitter equation for solution, let us try $x = y - 1$, and again $y^{-1} = u + 1/6$, giving $3888u^5 + 2160u^3 - 684u^2 + 1521u - 365 = 0$. Here the dominants are 3888, 1521, -365, so that the spans are a like quartic and an unlike linear; whence the smallest root is real positive, and the other four imaginary. For the imaginary roots let $u = (1521/3888)^{\frac{1}{4}} z = .79z$, giving $z^5 + .89z^3 - .36z^2 + z - .30 = 0$. Then $z^4 = -1 + 4(-.22z^2 + .09z + .08z^{-1})$, and $z = \omega + \omega^{-3}(-.22\omega^2 + .09\omega + .08\omega^{-1}) = -.08 + \omega - .09\omega^2 + .22\omega^3$. Here either $\omega = .707(1 \pm \sqrt{-1})$, $\omega^2 = \pm\sqrt{-1}$, $\omega^3 = .707(-1 \pm \sqrt{-1})$; or else $\omega = .707(-1 \pm \sqrt{-1})$, $\omega^2 = \mp\sqrt{-1}$, $\omega^3 = .707(1 \pm \sqrt{-1})$. For the first pair, $z = .47 \pm .8\sqrt{-1}$, and for the second, $z = -.63 \pm .9\sqrt{-1}$. The corresponding values of u are $.37 \pm .6\sqrt{-1}$ and $-.49 \pm .7\sqrt{-1}$. For the smallest root, let $u = 365v/1521 = .24v$, whence $.01v^5 + .08v^3 - .11v^2 + v - 1 = 0$, and $v = 1.02$, $u = .24$.

[5]. $x^4 - 2x^3 - 7x^2 + 10x + 10 = 0$. Dominants, 1, -7, final 10: two quadratics, each with unlike signs, hence two pairs of real roots, larger and smaller, the roots of each pair having opposite signs. If we put $x = y + \frac{1}{2}$, so that $y^4 - 17y^2/2 + 2y + 209/16 = 0$, the same characteristics appear. Even this will not give close approximations without going further in the series than we have hitherto done. We have in fact not yet had occasion to take more than the first term beyond ω, a term which I shall hereafter, overlooking the ω, speak of as the "first term," so as to speak of the term containing $(\phi E_0)^m$ as the m^{th} term. If for another trial, we take $x = s + 1$, and then $s^{-1} = t + 1/8$, clearing of fractions by $t = y/8$, so that $x = (y + 9)/(y + 1)$, we have $3y^4 - 130y^2 + 8y + 1159 = 0$. Let us take this as the equation to be solved, though it is really not much fitter than the one before. The characteristics are the same as for the original equation. For the larger pair, let $y = (130/3)^{\frac{1}{4}} u = 6.58\,u$, and let us carry the work to two places. We have $u^4 - u^2 + .0093\,u + .2057 = 0$, whence $u^2 = 1 + 2\phi u = 1 + 2\,(-.0047\,u^{-1} - .1029\,u^{-2})$. Then $(\phi u)^2 = .001\,u^{-3} + .011\,u^{-4}$, $(\phi u)^3 = -.001\,u^{-6}$. To the first term inclusive, the formula (14) is $\omega + \omega^{-1}\phi\omega = \omega + \omega^{-1}(-.005\,\omega^{-1} - .103\,\omega^{-2}) = -.005 + .897\omega$, where $\omega = \pm 1$, $\omega^2 = 1$. The second term is $\frac{1}{2}(\phi E_0)^2 \omega^{0-3}(0-1) = \frac{1}{2}(.001\,E_0^{-3} + .011\,E_0^{-4})\,\omega^{0-3}(0-1) = .0005\,\omega^{-6}(-4) + .0055\,\omega^{-7}(-5) = -.002 - .028\,\omega$. The third term is $\frac{1}{6}(\phi E_0)^3 \omega^{0-5}(0-1)(0-3) = \frac{1}{6}(-.001\,E_0^{-6})\,\omega^{0-5}(0-1)(0-3) = -.001/6\,\omega^{-11}(-7)(-9) = -.011\,\omega$. The second term gave us $.028\omega$, and the third now gives us $.011\omega$, both with the same negative sign, which must affect all further terms. As the convergency is not rapid, we may add $-.02\omega$ as probably covering the succeeding terms. Summing, $u = -.01 + .84\omega$, the two values of u being .83 and $-.85$, and from these the values of y are 5.5 and -5.6. For the remaining values of y, which must be smaller, let $y = (1159/130)^{\frac{1}{4}} v = 2.986\,v$, so that the equation becomes $.206\,v^4 - v^2 + .0206\,v + 1 = 0$, whence $v^2 = 1 + 2\,(.010\,v + .103\,v^4)$. Then $(\phi v)^2 = .002v^5 + .011\,v^6$, and $(\phi v)^3 = .001\,v^{10}$. The formula (14) gives $v = \omega + \omega^{-1}(.010\omega + .103\,\omega^4) + .004\omega^2 + .028\omega^3 + .011\,\omega^5 + \ldots$ Making as before a slight allowance for the terms neglected, again all of the same sign, say $+ .02\omega$, we have $v = .014 + 1.16\omega = .014 \pm 1.16$, and $y = 3v = .05 \pm 3.5$, so that there is a positive root somewhat greater than 3.5 and a negative root somewhat less. This equation, $3y^4 - 130y^2 + 8y + 1159 = 0$, may be resolved into the factors $3y^2 + 6y - 61 = 0$ and $y^2 - 2y - 19 = 0$.

[6]. $x^5 + 3x^4 + 2x^3 - 3x^2 - 2x - 2 = 0$. Nothing here is obviously domi-

nant. Let $x = y - 1$, so that $y^5 - 2y^4 - y^3 + 3y - 3 = 0$. For a fitter equation, let us take $y^{-1} = v + 1/5$; then $375v^5 - 25v^3 + 15v^2 + 256v - 1862/25 = 0$. This shows a like quartic and an unlike linear: smallest root real positive, the rest imaginary. For the imaginary roots, take $v = \frac{4}{5}(\frac{5}{3})^{\frac{1}{4}}u = .91u$; then $u^4 - (\frac{5}{3})^{\frac{1}{2}}u^2/16 + 3(\frac{5}{3})^{\frac{1}{4}}u/64 + 1 - 931(\frac{5}{3})^{-\frac{1}{4}}u^{-1}/2560 = 0$, and $u^4 = -1 + 4(.020u^2 - .015u + .080u^{-1})$. Hence $u = \omega + \omega^{-3}(.020\omega^2 - .015\omega + .080\omega^{-1}) = \omega - .080 + .015\omega^2 - .020\omega^3$, where $\omega^4 = -1$, and the values of ω and its powers are the same as in the paragraph numbered 4. Hence the imaginary values of u are approximately $.64 \pm .71\sqrt{-1}$ and $-.80 \pm .68\sqrt{-1}$, and those of v are $.58 \pm .64\sqrt{-1}$ and $-.73 \pm .62\sqrt{-1}$. For the small real root, take $v = 931z/3200 = .291z$, so that $z = 1 - .017z^2 + .008z^3 - .010z^5$. Hence $z = .98$ and $v = .29$.

[7]. $x^3 + 11x^2 - 102x + 181 = 0$. Dominants, 1, -102, 181: spans quadratic and linear, both unlike, hence three real roots, one of them negative, the smallest being positive. This is merely a form of the more celebrated equation $x^3 - 7x + 7 = 0$, to which it is reduced by writing $(x + 3)^{-1} + 3$ for x. The latter has the same characteristics, but may be made fitter by putting $x = 3y/2$, and $y^{-1} = 1 - \frac{1}{2}z^{-1}$, so that $x = 3z/(2z - 1)$ and $z^3 - 21z + 7 = 0$, again with the smallest root positive. To find the two larger roots, let $z = 21^{\frac{1}{2}}v = 4.583v$, whence $v^3 - v + .07274 = 0$, and $v^2 = 1 + 2(-.036)v^{-1}$. Taking up to the second term from the trinomial formula (2), we have $v = \omega - .036\omega^{-2} - .002\omega = -.036 + .998\omega$. As $\omega = \pm 1$, the values of v are -1.034 and $+.962$, whence the two larger values of z are -4.74 and $+4.41$. For the smallest root, let $z = \frac{1}{3}u$; whence $u = 1 + u^3/196$, and $u = 1.005$, $z = .335$.

[8]. $x^5 + x^4 + x^3 - 2x^2 + 2x - 1 = 0$. This may be a quintic with a quintic span, hence one real positive root, four imaginary. It is proper at this point to explain why there may be a dominant between the two at the ends. There are several considerations involved. As a rule, a variation of signs among the larger coefficients is favorable, because an accumulation of terms having the same sign is unfavorable to convergence. The most important point to examine, of course, is the absolute size of any doubtful coefficient. Other things equal, it is an unfavorable circumstance if any such coefficient is so large that it would be a dominant were the equation a trinomial; and we may therefore make a tentative use of the criterion of convergency for trinomials. To examine this case, let us suppose it reduced first to $x^5 - 2x^2 - 1 = 0$. Here $a = +\frac{2}{5}$, and the

case as a trinomial is convergent if, numerically, $a^n < k^{-k}(n-k)^{k-n}$, where $n=5$, $k=3$. We see that $2^5 5^{-5}$ is not less, but greater than $3^{-3}2^{-2}$. Next, suppose it reduced to $x^5+2x-1=0$. Here $a=-\frac{2}{5}$, $k=4$, and we see that $2^5 5^{-5} > 4^{-4}$. Both tests are contrary to convergence, and, notwithstanding the favorable unlikeness of the signs of -2 and 2, we may therefore suspect that the given equation contains two spans, a like quartic and an unlike linear. It may, however, be mentioned that a first approximation attempted from the quintic span gives $x=.68$, not far from exact. To obtain a fitter equation, take $x^{-1}=u+2/5$, so that $3125u^5+1250u^3+375u^2-3825u-4603=0$, this time undoubtedly a quintic span. Let $u=(4603/3125)^{1/5}v=1.08v$ nearly, so that $v^5=1+5(.18-.02v^2-.07v^3)$. Then $v=\omega+\omega^{-4}(.18-.02\omega^2-.07\omega^3)+\ldots.$, where $\omega^5=1$. Employing the powers of ω listed at the end of this paper, we find, for the real root, $v=1.09$, whence $u=1.17$; for one pair of imaginary roots, $v=.36\pm 1.2\sqrt{-1}$, and for the other pair, $v=-.91\pm .7\sqrt{-1}$, whence $u=.39\pm 1.3\sqrt{-1}$, $u=-.98\pm .8\sqrt{-1}$.

[9]. $x^6-6x^5-30x^2+12x-9=0$. Dominants, 1, -6, -30, -9; spans, an unlike linear, a like cubic, and a like quadratic. The largest root is real positive; the three next in size include one real negative and two imaginary having their real parts positive; the smallest pair is imaginary. For the large positive root, let $x=6u$, so that $u=1+5u^{-3}/216-u^{-4}/648+u^{-5}/5184$, whence without further approximation $u=1.02$, $x=6.12$. For the next three roots, let $x=5^{1/3}v=1.71v$, so that $v^3=-1+3(.10v^4+.08v^{-1}-.03v^{-2})$, whence $v=\omega+\omega^{-2}(.10\omega^4+.08\omega^{-1}-.03\omega^{-2})=-.08+\omega+.07\omega^2$. For $\omega=-1$, this gives $v=-1.01$ for the real root, and $x=-1.72$. For $\omega=\frac{1}{2}\pm .866\sqrt{-1}$, $\omega^2=-\frac{1}{2}\pm .866\sqrt{-1}$, it gives $v=.38\pm .92\sqrt{-1}$, $x=.6\pm 1.6\sqrt{-1}$. For the two smaller imaginary roots, let $x=(3/10)^{1/3}z=.548z$, so that $z^2=-1+2(.365z+.016z^5-.001z^6)$ or say $z^2=-1+2(.4z)$, whence $z=\omega+\omega^{-1}(.4\omega)$ $=.4+\omega=.4\pm\sqrt{-1}$, and $x=.2\pm .5\sqrt{-1}$.

[10]. $2x^6-18x^5+60x^4-120x^3-30x^2+18x-5=0$. Dominants, 2, -18, -120, -5: spans, an unlike linear, a like quadratic, and a like cubic. The largest root is therefore real positive, the next two in size are imaginary, and of the remaining three smaller roots, one is real negative and the other two imaginary with real parts positive. For the large positive root, let $x=9y$, so that $y=1-.37y^{-1}$, the rest being unimportant, whence $y=.63$, $x=5.6+$. For the next pair, let $x=(20/3)^{1/2}u=2.58u$, so that, disregarding the two final

terms, we have $u^2 = -1 + 2(.1u^3 + .7u - .1u^{-1})$, $u = \omega + \omega^{-1}(.1\omega^3 + .7\omega - .1\omega^{-1}) = .7 + \omega = .7 \pm \sqrt{-1}$, giving $x = 1.8 \pm 3\sqrt{-1}$. For the three smaller roots, let $x = 24^{-1/3}v = .347v$, so that, neglecting the two first terms, $v^3 = -1 + 3(.4v - .2v^2 + .1v^4)$, $v = \omega + \omega^{-2}(.4\omega - .2\omega^2 + .1\omega^4) = -.2 + \omega - .3\omega^2$. For the real root, $\omega = -1$, $v = -1.5$, $x = -.5$. For the imaginary, $\omega = .5 \pm .866\sqrt{-1}$, $\omega^2 = -.5 \pm .866\sqrt{-1}$, and $v = .5 \pm .6\sqrt{-1}$, $x = .2 \pm .2\sqrt{-1}$.

[11]. $2x^3 + 15x^2 - 84x - 190 = 0$. Dominants, 2, -84, -190: a quadratic span, with unlike signs, and a linear, with like signs. There are therefore three real roots, two of them negative and one of these the smallest, the third positive. If $x = y - 5/2$, we have the equation heretofore fully discussed as (3).

Lest the reader may suppose that, in the examples above illustrated, unusual means have been employed to secure specially favorable transformations, it is proper to remark that, while such unusual means ought to be discovered for the most perfect working of the method, none worth mentioning has yet been devised. The transformations used have, with one exception, been of the simplest, as the following list will show. In this list the word "direct" means "direct suppression" of the second term, and "reverse" means "reverse suppression" of the term next to the last. Newton's equation, reverse; Murphy's, none; Jelinek's, none; equation (3), none; No. 1, $x = y - 1$; No. 2, $x = 2y$, $y = (1 + u)/(1 - u)$; No. 3, $x^{-1} = 1 - y$, and reverse; No. 4, $x = y - 1$, and reverse; No. 5, first transformation, direct; second, $x = s + 1$, and reverse; No. 6, $x = y - 1$, and reverse; No. 7, from the usual form, $x = \tfrac{2}{3}y$, and reverse; No. 8, reverse; No. 9, none; No. 10, none; No. 11, direct. I am able to make only one suggestion, illustrated by No. 2. If an equation $f(x) = 0$ can be brought to the form in which $f(1) = \pm 1$, the transformation $x = (1 + u)/(1 - u)$ may be valuable. No. 2 is $x^4 - 4x^3 - 3x + 23 = 0$, and it is put into the form in question by taking $x = 2y$, whence $16y^4 - 32y^3 - 6y + 23 = 0$.

The present method applies equally well to the computation of the roots of equations with imaginary coefficients, coefficients which are regarded as having respectively the size of their moduli. Such equations are of greater generality than the ordinary, in which the imaginary roots are restricted to going in pairs. For a simple example, take $x^3 - x\sqrt{-1} - 1 = 0$. Here the span is cubic, $\omega^n = 1$, $a = \tfrac{1}{3}\sqrt{-1}$, and the formula from (2) is $x = \omega + \omega^2 a - \tfrac{1}{3}\omega a^3 + \tfrac{1}{3}\omega^2 a^4 + \ldots$. Here $a^3 = -a/9$, $a^4 = 1/81$, so that $x = \omega + \omega^2/243 + a(\omega^2 + \omega/27)$, disregarding the subsequent terms. As either $\omega = 1$, $\omega^2 = 1$, or else $\omega = -\tfrac{1}{2} \pm \sqrt{-3}$,

$\omega^3 = -\frac{1}{2} \mp \sqrt{-3}$, we find directly the three values of x, namely, $1.004 + .346\sqrt{-1}$, $-.225 + .690\sqrt{-1}$, and $-.779 - 1.036\sqrt{-1}$, which vary but slightly, if at all, from the accurate values. When the independent term ω^n is imaginary, the values of ω may be found by the usual method for computing the n^{th} roots of an imaginary quantity by the aid of De Moivre's theorem.

That difficulties will arise when we attempt to apply the formula to cases in which there are no obvious dominants is certain. The case of equal roots has already been mentioned as of that nature. Equal roots may be regarded as on the line between the real and the imaginary, and so we may suppose that an equation containing them may, as to its characteristics, be exactly on the dividing line between one set of dominants and another. Even a probably convergent case may present difficulties when two roots are equal: for example, $x^5 + x^4 - x - 1 = 0$. Treating this as a quintic span, we must accept the sum of two of the four imaginary series produced (if they can be summed, as seems probable), as equal to zero. To illustrate another class of doubtful cases, in which one or more of the series is divergent, let us take $x^4 + x^3 - x^2 + x + 1 = 0$. If we treat this as a quartic span we shall derive for the imaginary roots $.65 \pm .76\sqrt{-1}$, but for the real roots we shall obtain, instead of $-1.15 \pm .57$, the unintelligible result $-1.15 - \infty\sqrt{-1}$. If the sign of x be changed and 1 substituted for x we have as the sum of the terms -1, and we may put $x = -(1 + u)/(1 - u) = (u + 1)/(u - 1)$, producing $3u^4 + 14u^2 - 1 = 0$, an equation eminently fit. In treating this equation as a quartic span we were in fact assuming it to have four imaginary roots. The result shows that we should look upon it as having spans either linear and cubic; linear, quadratic and linear, or cubic and linear.

Having thus a method for the easy determination of the values of the roots, real and imaginary, not to speak of the preliminary recognition of their nature by inspection, there seems to be no further need of those various methods for distinguishing the real roots and assigning limits to their values, of which Sturm's was the latest and best. Are we then to say that for computing more closely the value of a single real root we may also dispense with the orderly and satisfactory method which we owe to Horner? The answer may eventually be both yes and no. As the culmination of arithmetical processes, based on the same principles as other well-known and less general rules of procedure, such for instance as the extraction of the square root of a given number, it is not supposable that Horner's method can ever be allowed to subside into obscurity. On the other hand, apart from the orderly and luminous character of the process itself, it has never been really proved to be intrinsically easier in actual work than that by series, due to Euler and Lagrange, a process which, as we have seen, is only a special case of the present general method, that case in which $n = 1$. It may be seriously questioned whether, for practical purposes, any one employing the present method for first acquaintance with the several roots will, unless for special reasons in special cases, abandon it for Horner's method when it comes to securing a closer approximation to the value of any one of them. To use the present method in such a case, having an approximate value a, it is only necessary to transform the equation by $x = y + a$, and then, having thus secured a linear span for the smallest root, make another application of the method by the use of formula (14). De Morgan once deprecated a suggested improvement of Horner's method, "considering that the process is one which no person will very often perform," since variations of known rules infrequently applied "afford greater assistance in forgetting the method than in abbreviating it." For a similar reason it may not unreasonably be assumed that the present method may probably, in practice, be followed in most cases to the end.*

Sometimes, indeed, the examples employed for illustrating Horner's method are such as the present method meets at once most satisfactorily, without requiring further treatment at all. For example, Todhunter's chief example, running over many pages, is $x^3 - 3x^2 - 2x + 5 = 0$, which, if $x = y + 1$ (direct suppression), is $y^3 - 5y + 1 = 0$, a quadratic and a linear span yielding for all the roots very convergent results by the use of (2). Again, the last example given by Burnside and Panton is $x^4 - 3x^2 + 75x - 10000 = 0$, as to which it is required to "find to three places of decimals the root situated between 9 and 10." The result is 9.886. If we put $x = 10y$, we have a quartic span, $y^4 = 1 + 4\phi y$, where $\phi y = .0075\,y^2 - .01875\,y$. Taking nothing beyond the second term of the general series (14), we have readily, for the four values of y, .9886, —1.0261, and $.01875 \pm .9927\sqrt{-1}$.

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*It will of course have been observed that, at least for determining the numerical value of ω when ω^n is not ± 1, or (what is the same) for transforming the equation to the form $\omega^n = \pm 1$, the present method is much facilitated by the use of logarithms. It is not inappropriate to quote from Burnside and Panton the quasi-prediction with which they close their book, at the end of a historical note on numerical equations: "Mathematicians may also invent, in process of time, some mode of calculation applicable to numerical equations analogous to the logarithmic calculation of simple roots. But at the present time the most perfect solution of Lagrange's problem is to be sought in a combination of the methods of Sturm and Horner."

For the elementary instruction of students it may be well to afford a first glimpse of the method of dominants by taking some simple case in which there are only two or three large coefficients, with the other coefficients relatively much smaller. For instance, the cubic $1000x^3 + x - 1000 = 0$ will be admitted by any one to have three roots not very far in value from the three cube roots of unity; or if it be disputed, the discrepancy can be increased till it be admitted. Then $x^3 + x^2 + 10000x - 1 = 0$ may be used in like manner. Any one can easily be made to see that this resembles, first, a quadratic, and secondly, a linear equation. For, if $x = 100y$, we have $y^2 + y/100 + 1 - y^{-1}/1000000 = 0$, in which the last term is plainly of little importance; and similarly, if $x = u/10000$, we have $u = 1 - u^2/100000000 - u^3/1000000000000$, which cannot differ much from $u = 1$.

A simple proof of the convergency-criterion stated for trinomials is had by regarding the quotient formed by dividing the general $(m+n)^{\text{th}}$ term by the m^{th}, and considering m increased indefinitely. This supplies the criterion for a series comprised of the terms numbered m, $m+n$, $m+2n$, and so on, and the general series comprises n series of that nature, all having the same criterion. We have seen, as regards trinomials in which the middle term is not a dominant, that all the roots can be found by a single convergent series; and when the middle term is a dominant, which means, when the middle term has any value other than those included in the first case, the n roots are broken up into two classes, k roots of the first class being found by one convergent series, and $n-k$ roots of the second class being found by a second convergent series. All this is proved by the aid of the criterion of convergency. That similar facts are true concerning equations having more than three terms is also reasonably clear, yet in the absence of a general convergency-criterion we have not the same sort of proof for such equations as we have for trinomials, that every equation must be composed of mutually exclusive spans for each of which the series is convergent.

That the nature of the roots, as to reality and relative size, is disclosed when fitness is established and the spans recognized, is sufficiently proved. Our inferences as to the signs of the roots depend on an assumption not yet proved, viz. that the sign of the sum of the series is the same as that of the ω with which the series begins. No such doubt exists concerning the distinction between real and imaginary roots. The imaginary series always occur in pairs of equal value and of opposite signs. If, in any case wherein ω is imaginary, the sum of the imaginary terms is not zero, there are two imaginary roots, while if it is zero there are two equal roots. Notwithstanding the defect of proof as yet existing concerning the signs of the roots, the inferences derivable from recognized spans are most of them certain and all of them tentatively useful. When we subsequently ascertain by convergent series all the roots of the equation, the proof that the values so found are really the roots is absolute, and the correctness of the preliminary inferences made as to their nature is determined.

The following table shows the maximum value of a^n consistent with convergency for any trinomial coming within the limits of the table, and may be extended by reference to the criterion $a^n < k^{-k}(n-k)^{k-n}\omega^{nk}$. When ω^n has any other value than ± 1, the factor ω^{nk} must be supplied.

$n-k$.	$n=1$.	$n=2$.	$n=3$.	$n=4$.	$n=5$.	$n=6$.
8	$7^7.8^{-8}$	$6^6.8^{-8}$	$5^5.8^{-8}$	$4^4.8^{-8}$	$3^3.8^{-8}$	$2^2.8^{-8}$
7	$6^6.7^{-7}$	$5^5.7^{-7}$	$4^4.7^{-7}$	$3^3.7^{-7}$	$2^2.7^{-7}$	7^{-7}
6	$5^5.6^{-6}$	$4^4.6^{-6}$	$3^3.6^{-6}$	$2^2.6^{-6}$	6^{-6}	
5	$4^4.5^{-5}$	$3^3.5^{-5}$	$2^2.5^{-5}$	5^{-5}		5^{-5}
4	$3^3.4^{-4}$	$2^2.4^{-4}$	4^{-4}		4^{-4}	$4^{-4}.2^{-2}$
3	$2^2.3^{-3}$	3^{-3}		3^{-3}	$3^{-3}.2^{-2}$	$3^{-3}.3^{-3}$
2	2^{-2}		2^{-2}	$2^{-2}.2^{-2}$	$2^{-2}.3^{-3}$	$2^{-2}.4^{-4}$
1		1	2^{-2}	3^{-3}	4^{-4}	5^{-5}
0						
—1	2^{-2}	3^{-3}	4^{-4}	5^{-5}	6^{-6}	7^{-7}
—2	$2^2.3^{-3}$	$2^2.4^{-4}$	$2^2.5^{-5}$	$2^2.6^{-6}$	$2^2.7^{-7}$	$2^2.8^{-8}$
—3	$3^3.4^{-4}$	$3^3.5^{-5}$	$3^3.6^{-6}$	$3^3.7^{-7}$	$3^3.8^{-8}$	$3^3.9^{-9}$
—4	$4^4.5^{-5}$	$4^4.6^{-6}$	$4^4.7^{-7}$	$4^4.8^{-8}$	$4^4.9^{-9}$	$4^4.10^{-10}$
—5	$5^5.6^{-6}$	$5^5.7^{-7}$	$5^5.8^{-8}$	$5^5.9^{-9}$	$5^5.10^{-10}$	$5^5.11^{-11}$

The following list of certain roots of 1 and of —1, and of the powers of these roots, will be found useful for reference. If $\omega^2=1$, $\omega=\pm 1$. If $\omega^2=-1$, $\omega=\pm\sqrt{-1}$. If $\omega^3=1$, $\omega=1$, $\omega^2=1$, or else $\omega=-\frac{1}{2}\pm\frac{1}{2}\sqrt{-3}$, $\omega^2=-\frac{1}{2}\mp\frac{1}{2}\sqrt{-3}$. If $\omega^3=-1$, $\omega=-1$, $\omega^2=1$, or else $\omega=\frac{1}{2}\pm\frac{1}{2}\sqrt{-3}$, $\omega^2=-\frac{1}{2}\pm\frac{1}{2}\sqrt{-3}$. If $\omega^4=1$, either $\omega=1$, $\omega^2=1$, $\omega^3=1$; or $\omega=-1$, $\omega^2=1$, $\omega^3=-1$; or $\omega=\pm\sqrt{-1}$, $\omega^2=-1$, $\omega^3=\mp\sqrt{-1}$. If $\omega^4=-1$, either $\omega=\sqrt{\frac{1}{2}}(1\pm\sqrt{-1})$, $\omega^2=\pm\sqrt{-1}$,

$\omega^3 = \sqrt{\tfrac{1}{2}}(-1 \pm \sqrt{-1})$; or $\omega = \sqrt{\tfrac{1}{2}}(-1 \pm \sqrt{-1})$, $\omega^2 = \mp\sqrt{-1}$, $\omega^3 = \sqrt{\tfrac{1}{2}}(1 \pm \sqrt{-1})$. If $\omega^5 = 1$, either $\omega = 1$, $\omega^2 = 1$, $\omega^3 = 1$, $\omega^4 = 1$; or $\omega = .309 \pm .951\sqrt{-1}$, $\omega^2 = -.809 \pm .587\sqrt{-1}$, $\omega^3 = -.809 \mp .587\sqrt{-1}$, $\omega^4 = .309 \mp .951\sqrt{-1}$; or $\omega = -.809 \pm .587\sqrt{-1}$, $\omega^2 = .309 \mp .951\sqrt{-1}$, $\omega^3 = .309 \pm .951\sqrt{-1}$, $\omega^4 = -.809 \mp .587\sqrt{-1}$. (The fractions .309, .809, .587, .951, are given merely as approximations to the values of the exact expressions $\tfrac{1}{4}(\sqrt{5}-1)$, $\tfrac{1}{4}(\sqrt{5}+1)$, $\tfrac{1}{4}\sqrt{(10-2\sqrt{5})}$, and $\tfrac{1}{4}\sqrt{(10+2\sqrt{5})}$ respectively.) If $\omega^5 = -1$, either $\omega = -1$, $\omega^2 = 1$, $\omega^3 = -1$, $\omega^4 = 1$; or $\omega = -.309 \pm .951\sqrt{-1}$, $\omega^2 = -.809 \mp .587\sqrt{-1}$, $\omega^3 = .809 \mp .587\sqrt{-1}$, $\omega^4 = .309 \pm .951\sqrt{-1}$; or $\omega = .809 \pm .587\sqrt{-1}$, $\omega^2 = .309 \pm .951\sqrt{-1}$, $\omega^3 = -.309 \pm .951\sqrt{-1}$, $\omega^4 = -.809 \pm .587\sqrt{-1}$. If $\omega^6 = -1$, see the example at the beginning. The list may readily be extended by the rules given in the text-books.

Sur le logarithme de la fonction gamma.

PAR CH. HERMITE.

Je vais revenir encore à l'intégrale de Raabe pour présenter son rôle sous un nouveau jour, en considérant l'expression plus générale,

$$\tfrac{1}{2}\log\left[\Gamma(a+\xi)\,\Gamma(a+1-\xi)\right],$$

et montrant qu'elle en donne la valeur asymptotique sous la condition que ξ soit positif et moindre que l'unité. Ce résultat a été déja établi dans un article sur l'extension de la formule de Stirling (Mathematischen Annalen, T. 41, p. 581), on va voir qu'on y parvient plus facilement par la nouvelle méthode que je vais indiquer.

Soit $F(x)$ une fonction qui ne change pas lorsqu'on y remplace x par $1-x$, je partirai de l'égalité suivante,

$$\int_0^{\xi} xF'(x)\,dx+\int_0^{1-\xi} xF'(x)\,dx=F(\xi)-\int_0^1 F(x)\,dx,$$

qui se vérifie immédiatement en observant que l'on a,

$$\int_0^{\xi} xF'(x)\,dx=\xi F(\xi)-\int_0^{\xi} F(x)\,dx,$$

puis,

$$\int_0^{\xi} F(x)\,dx+\int_0^{1-\xi} F(x)\,dx=\int_0^1 F(x)\,dx,$$

sous la condition admise à l'égard de $F(x)$.

Prenons maintenant,

$$F(x)=\log\left[\Gamma(a+x)\,\Gamma(a+1-x)\right],$$

en désignant par J l'intégrale de Raabe, on aura, évidemment,

$$2J=\int_0^1 \log\left[\Gamma(a+x)\,\Gamma(a+1-x)\right]dx.$$

Soit ensuite,

$$J_1(a) = \tfrac{1}{2}\int_0^{\xi} x D_x \log\left[\Gamma(a+x)\,\Gamma(a+1-x)\right] dx$$
$$+ \tfrac{1}{2}\int_0^{1-\xi} x D_x \log\left[\Gamma(a+x)\,\Gamma(a+1-x)\right] dx,$$

nous en conclurons cette relation,

$$\tfrac{1}{2}\log\left[\Gamma(a+x)\,\Gamma(a+1-x)\right] = J + J_1(a),$$

et le résultat annoncé sera mis en évidence au moyen d'une expression de $J_1(a)$ que je vais obtenir.

J'observe à cet effet que de la formule,

$$D_x \log\Gamma(x) = \int_{-\infty}^{0}\left[\frac{e^{xy}}{e^y-1} - \frac{e^y}{y}\right] dy,$$

on tire aisement,

$$D_x \log\left[\Gamma(a+x)\,\Gamma(a+1-x)\right] dx = \int_{-\infty}^{0} \frac{\left[e^{xy}-e^{(1-x)y}\right]e^{ay}}{e^y-1}\, dy,$$

il en résulte qu'on peut écrire en désignant par y_0 une certaine valeur de la variable qui dépend de a,

$$\int_{-\infty}^{0} \frac{\left[e^{xy}-e^{(1-x)y}\right]e^{ay}}{e^y-1}\, dy = \frac{e^{xy_0}-e^{(1-x)y_0}}{e^{y_0}-1}\int_{-\infty}^{0} e^{ay}\, dy = \frac{e^{xy_0}-e^{(1-x)y_0}}{(e^{y_0}-1)\,a}.$$

Cela posé, cherchons une limite supérieure de la fonction $\dfrac{e^{xy}-e^{(1-x)y}}{e^y-1}$, et dans ce but, mettons la sous la forme,

$$\frac{e^{(2x-1)\frac{y}{2}} - e^{-(2x-1)\cdot\frac{y}{2}}}{e^{\frac{y}{2}} - e^{-\frac{y}{2}}}.$$

En developpant en série, elle devient,

$$\frac{(2x-1)\,y + \dfrac{1}{2.3.2^2}\left[(2x-1)\,y\right]^3 + \dfrac{1}{2.3.4.5.2^4}\left[(2x-1)\,y\right]^5 + \ldots\ldots}{y + \dfrac{1}{2.3.2^2}\,y^3 + \dfrac{1}{2.3.4.5.2^4}\,y^5 + \ldots\ldots},$$

c'est-à-dire,

$$(2x-1)\,\frac{1 + \dfrac{1}{2.3.2^2}\left[(2x-1)\,y\right]^2 + \dfrac{1}{2.3.4.5.2^4}\left[(2x-1)\,y\right]^4 + \ldots\ldots}{1 + \dfrac{1}{2.3.2^2}\,y^2 + \dfrac{1}{2.3.4.5.2^4}\,y^4 + \ldots\ldots}.$$

On voit que si l'on suppose x compris entre zéro et l'unité le numérateur de la quantité qui multiplie $2x-1$, est toujours moindre que le dénominateur; il en résulte qu'en faisant,

$$\frac{e^{xy}-e^{(1-x)y}}{e^{y}-1}=(2x-1)\phi(x),$$

la fonction $\phi(x)$ sera positive, moindre que l'unité pour toutes les valeurs réelles de la variable y et par conséquent de a; on aura aussi la condition $\phi(x)=\phi(1-x)$.

Cela étant l'expression de $J_1(a)$ prend cette nouvelle forme,

$$J_1(a)=\frac{1}{2a}\int_0^{\xi}x(2x-1)\phi(x)\,dx+\frac{1}{2a}\int_0^{1-\xi}x(2x-1)\phi(x)\,dx,$$

ou bien, $J_1(a)=\frac{M}{2a}$ qui suffit à notre objet, la quantité,

$$M=\int_0^{\xi}x(2x-1)\phi(x)\,dx+\int_0^{1-\xi}x(2x-1)\phi(x)\,dx,$$

étant finie évidemment quel que soit a. Mais nous irons plus loin en obtenant les limites indépendantes de a entre lesquelles elle reste toujours comprise.

J'employe pour cela la relation générale,

$$\int_0^{\xi}f(x)\,dx+\int_0^{1-\xi}f(x)\,dx=\int_0^{\xi}[f(x)-f(1-x)]\,dx+\int_0^{1}f(x)\,dx,$$

qui se vérifie en remarquant que par le changement de x en $1-x$, on trouve,

$$\int_0^{1-\xi}f(x)\,dx=\int_0^{1}f(x)\,dx-\int_0^{\xi}f(1-x)\,dx,$$

ou plutôt encore celle-ci,

$$\int_0^{\xi}f(x)\,dx+\int_0^{1-\xi}f(x)=\int_0^{\xi}[f(x)-f(1-x)]\,dx+\tfrac{1}{2}\int_0^{1}[f(x)+f(1-x)]\,dx.$$

Cela étant, soit,

$$f(x)=x(2x-1)\phi(x);$$

de la propriété qui a été indiquée tout-à-l'heure de la fonction $\phi(x)$, on tire,

$$\begin{aligned}f(x)-f(1-x)&=-(1-2x)\,\phi(x),\\ f(x)+f(1-x)&=(1-2x)^2\phi(x)\end{aligned}$$

et nous avons en conséquence:

$$M=\tfrac{1}{2}\int_0^{1}(1-2x)^2\phi(x)\,dx-\int_0^{\xi}(1-2x)\phi(x)\,dx.$$

La première intégrale en se rappelant qu'on a $\phi(x) < 1$ s'exprime par $\frac{1}{2}\int_0^1 (1-2x)^2 dx.\theta = \frac{\theta}{6}$, θ étant moindre que l'unité, la seconde si l'on suppose comme on peut le faire, $\xi < \frac{1}{2}$, aura pour valeur $\int_0^{\xi} (1-2x)\,dx.\theta = (\xi - \xi^2)\theta'$, θ' étant aussi compris entre zéro et un, nous avons donc ce résultat qu'il s'agissait d'obtenir,

$$M = \frac{\theta}{6} - (\xi - \xi^2)\theta'.$$

Soit en particulier $\xi = 1$, ce qui donne $M = \frac{\theta}{6}$, on trouve alors la relation,

$$\tfrac{1}{2}\log[\Gamma(a+1)\Gamma(a)] = J + \frac{\theta}{12a},$$

d'où l'on conclut,

$$\log\Gamma(a) = J - \tfrac{1}{2}\log a + \frac{\theta}{12a}.$$

C'est l'expression asymptotique à laquelle j'étais parvenu précédemment par une autre méthode. La quantité $J_1(a)$ représentée par l'intégrale

$$\tfrac{1}{2}\int_0^1 x D_x \log[\Gamma(a+x)\Gamma(a+1-x)]\,dx,$$

coïncide dans ce cas avec $J(a) = \frac{1}{2}\int_0^1 (x-x^2) D_x^2 \log\Gamma(a+x)\,dx$, voici comment on passe de la première forme à la seconde. Changeons x en $1-x$, nous aurons d'abord,

$$J_1(a) = -\tfrac{1}{2}\int_0^1 (1-x) D_x \log[\Gamma(a+x)\Gamma(a+1-x)]\,dx,$$

en ajoutant membre à membre avec la valeur précédente il vient

$$J_1(a) = \tfrac{1}{2}\int_0^1 (2x-1) D_x \log[\Gamma(a+x)\Gamma(a+1-x)]\,dx.$$

Le facteur $2x-1$ étant la dérivée de x^2-x, une intégration par parties donne facilement, si l'on observe que $x-x^2$ ne change pas lorsqu'on remplace x par $1-x$,

$$\begin{aligned} 2J_1(a) &= \tfrac{1}{2}\int_0^1 (x-x^2) D_x^2 \log[\Gamma(a+x)\Gamma(a+1-x)]\,dx \\ &= \int_0^1 (x-x^2) D_x^2 \log\Gamma(a+x)\,dx. \end{aligned}$$

Soit ensuite $\xi = \frac{1}{2}$, on trouve $M = \frac{\theta}{6} - \frac{\theta'}{4}$; il est nécessaire alors pour avoir une limite plus précise, de recourir à l'expression générale,

$$M = \int_0^{\xi} x(2x-1)\,\phi(x)\,dx + \int_0^{1-\xi} x(2x-1)\,\phi(x)\,dx,$$

d'où l'on tire,

$$\begin{aligned} M &= 2\int_0^{\frac{1}{2}} x(2x-1)\,\phi(x)\,dx \\ &= -2\int_0^{\frac{1}{2}} x(1-2x)\,\phi(x)\,dx. \end{aligned}$$

Le facteur $x(1-2x)$ étant positif entre les limites de l'intégrale et $\phi(x)$ étant moindre que l'unité, nous avons cette valeur,

$$\begin{aligned} M &= -2\int_0^{\frac{1}{2}} x(1-2x)\,dx\,.\,\theta \\ &= -\frac{\theta}{12} \end{aligned}$$

et l'on en conclut l'expression asymptotique,

$$\log \Gamma\left(a + \tfrac{1}{2}\right) = J - \frac{\theta}{24a}.$$

Je reviens maintenant à la formule générale,

$$\begin{aligned} J_1(a) &= \tfrac{1}{2}\int_0^{\xi} x D_x \log\left[\Gamma(a+x)\,\Gamma(a+1-x)\right] dx \\ &+ \tfrac{1}{2}\int_0^{1-\xi} x D_x \log\left[\Gamma(a+x)\,\Gamma(a+1-x)\right] dx, \end{aligned}$$

afin d'en tirer une autre expression de $J_1(a)$ qui permet d'obtenir son développement en série suivant les puissances descendantes de a. Ce résultat important se déduit aisément de l'égalité,

$$D_x \log\left[\Gamma(a+x)\,\Gamma(a+1-x)\right] = \int_{-\infty}^{\infty} \frac{\left[e^{xy} - e^{(1-x)y}\right] e^{ay}}{e^y - 1}\,dy,$$

au moyen des intégrales suivantes,

$$\int_0^{\xi} \left[e^{xy} - e^{(1-x)y}\right] x\,dx = e^{\xi y}\left(\frac{\xi}{y} - \frac{1}{y^2}\right) + e^{(1-\xi)y}\left(\frac{\xi}{y} + \frac{1}{y^2}\right) - \frac{e^y - 1}{y^2},$$

$$\int_0^{1-\xi} \left[e^{xy} - e^{(1-x)y}\right] x\,dx = e^{(1-\xi)y}\left(\frac{1-\xi}{y} - \frac{1}{y^2}\right) + e^{\xi y}\left(\frac{1-\xi}{y} + \frac{1}{y^2}\right) - \frac{e^y - 1}{y^2}.$$

En les ajoutant on trouve la quantité $\frac{e^{\xi y} + e^{(1-\xi)y}}{y} - 2\,\frac{e^y - 1}{y^2}$ ce qui donne immédiatement,

$$J_1(a) = \int_{-\infty}^{0} \left[\frac{e^{\xi y} + e^{(1-\xi)y}}{2y(e^y - 1)} - \frac{1}{y^2}\right] e^{ay} dy.$$

Supposons $\xi = 1$, on en tire la formule de Stirling,

$$J_1(a) = \frac{B_1}{1.2.a} - \frac{B_2}{3.4.a^3} + \frac{B_3}{5.6.a^5} - \dots\,,$$

où B_1, B_2, etc., désignent suivant l'usage les nombres de Bernouilli. En faisant $\xi = \frac{1}{2}$ on en conclut la série de Gauss,

$$J_1(a) = -\frac{B_1}{1.2.\,2a} + \frac{(2^3 - 1)B_2}{3.4.\,2^3.a^3} - \frac{(2^5 - 1)B_3}{5.6.\,2^5.a^5} + \dots\,,$$

ce second développement pouvant se deduire du premier au moyen de la relation,

$$J_1(a) = J(2a) - J(a),$$

qui découle facilement des expressions,

$$J(a) = \int_{-\infty}^{0} \left[\frac{e^y + 1}{2y(e^y - 1)} - \frac{1}{y^2}\right] e^{ay} dy,$$

$$J(2a) = \int_{-\infty}^{0} \left[\frac{e^y + 1}{2y(e^y - 1)} - \frac{1}{y^2}\right] e^{2ay} dy.$$

Remplaçons en effet dans la seconde y par $\frac{y}{2}$ et retranchons membre à membre, il vient après une réduction évidente,

$$J(2a) - J(a) = \int_{-\infty}^{0} \left[\frac{e^{\frac{1}{2}y}}{y(e^y - 1)} - \frac{1}{y^2}\right] e^{ay} dy,$$

c'est-à-dire la valeur de $J_1(a)$ pour $\xi = \frac{1}{2}$. Dans le cas général où ξ est quelconque, je rappelle en terminant que si l'on désigne par $s_n(\xi)$, le polynôme de degré $n + 1$ de Jacob Bernouilli, égal lorsque ξ est entier à la somme, $1^n + 2^n + \dots + (\xi - 1)^n$ on a la série suivante,

$$J_1(a) = \sum \left[\frac{(-1)^{n-1}B_n}{n(2n - 1)} + \frac{2S_{2n-1}(\xi)}{2n - 1}\right] \frac{1}{a^{2n-1}}. \qquad (n = 1, 2, 3, \dots)$$

On trouvera dans l'article des Mathematischen Annalen qui a été cité plus haut, la démonstration de cette formule et les conditions de son emploi.

Sur la pression dans les milieux diélectriques ou magnétiques.

PAR P. DUHEM, *Professeur à la Faculté des Sciences de Bordeaux.*

INTRODUCTION.

Tous les physiciens connaissent la théorie des pressions dans les milieux polarisés qu'a imaginée Maxwell; perfectionnée par H. von Helmholtz, par G. Kirchhoff, par M. E. Lorberg, cette théorie ne peut éviter des difficultés et des contradictions qui ont été signalées par M. Beltrami, par E. Mathieu, par M. M. Brillonin.

Nous avons tenté, dans nos *Leçons sur l'Electricité et le Magnétisme*, de donner une théorie des pressions dans les milieux magnétiques ou diélectriques, distincte de celle que Maxwell a indiquée et exempte des difficultés que cette dernière présente; la méthode que nous avons suivie, fondée sur l'emploi du principe des vitesses virtuelles, paraît hors de contestation; malheureusement, une erreur s'est glissée dans nos calculs; en estimant la variation virtuelle du potentiel magnétique d'un système, nous avons négligé, comme infiniment petit du second ordre, une quantité qui était en réalité un infiniment petit du premier ordre; il résulte de cette erreur que les conditions d'équilibre qui se réfèrent aux divers points de la surface limite d'un milieu polarisé ont été données par nous d'une manière inexacte; au contraire, les conditions d'équilibre qui se réfèrent aux points situés à l'intérieur de la masse magnétique ou diélectrique ont été exactement données. La même erreur entache les conditions aux limites que nous avons établies en étudiant les solutions d'un sel magnétique.*

Cette erreur a été signalée par M. Liénard.† M. Liénard a donné une évaluation du terme négligé par nous; cette évaluation le conduit à une conséquence remarquable: pour maintenir en équilibre un fluide polarisé, il faut appliquer à chacun des éléments de la surface qui le limite une pression dont la direction est normale à l'élément, mais *dont la grandeur dépend de l'orientation de l'élément;* la grandeur de cette pression en un point de l'élément est $2\pi\varepsilon M^2 \cos^2(M, N_i)$, ε étant la constante des lois de Coulomb et M l'intensité de polarisation.

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* Sur les dissolutions d'un sel magnétique (Annales de l'Ecole Normale Sup^{e}. 3^{e} série, t. VII, p. 289. 1890).

† Liénard, *Pressions à l'intérieur des aimants et des diélectriques.* (La lumière électrique. Tome LII, p. 7 et p. 67, 1894.)

Lorsque le corps est assez faiblement polarisé pour que l'on puisse négliger son potentiel sur lui même, cette pression introduite par M. Liénard devient proportionnelle au carré de l'intensité du champ et *au carré du coefficient de polarisation* du corps; au contraire, tous les autres termes que la polarisation conduit à introduire dans l'étude des pressions sont proportionnels au carré de l'intensité du champ et *à la première puissance du coefficient de polarisation;* le terme complémentaire introduit par M. Liénard peut donc être négligé lorsque l'on considère des corps faiblement diélectriques ou faiblement magnétiques; pour de tels corps, la théorie que nous avons donnée subsiste en entier. Au contraire, pour les corps fortement magnétiques tels que le fer doux, le terme complémentaire a une grande valeur.

Les belles recherches de M. Liénard nous ont amené à reprendre, à notre tour, l'étude des pressions dans les milieux polarisés; cette étude repose, comme du reste la mise en équation de tous les problèmes relatifs aux corps magnétiques ou diélectriques, sur l'expression de la variation infiniment petite qu'éprouve le potentiel d'un système polarisé sur lui même, lorsque ce système éprouve une modification infiniment petite; cette expression, qui était incomplète dans nos *Leçons sur l'Electricité et le Magnétisme*, nous avons cherché à l'établir avec rigueur.

La méthode qui sert à traiter avec précision les questions relatives à la fonction potentiel ou au potentiel d'un *système polarisé*, où A, B, C, sont les composantes de la polarisation au point (x, y, z), est bien connue; elle consiste à ramener, au moyen d'intégrations par parties, la question proposée à une question analogue relative à un *système électrisé*, portant, en chaque point de sa masse, une densité électrique solide,

$$\rho = -\left(\frac{\partial A}{\partial x} + \frac{\partial B}{\partial y} + \frac{\partial C}{\partial z}\right) \tag{1}$$

et, en chaque point de sa surface, une densité électrique superficielle,

$$\sigma = -\left[A \cos(N_i, x) + B \cos(N_i, y) + C \cos(N_i, z)\right]. \tag{2}$$

Cette méthode ramène la question que nous nous étions proposée à celle-ci;

trouver l'expression de la variation infiniment petite qu'éprouve le potentiel électrostatique d'un système lorsque la forme, la position et l'électrisation de ce système éprouvent des modifications infiniment petites. Ce dernier problème peut, d'ailleurs, être regardé comme le problème fondamental de l'électrostatique; il serait donc désirable que la solution en soit donnée avec la rigueur que Gauss, Bouquet et M. O. Holder ont apporté dans les démonstrations relatives à la fonction potentielle; cette solution, qui n'avait jamais été donnée à notre connaissance, est l'objet du Chapître I du présent Mémoire.

Au Chapître II, nous avons montré brièvement comment on pouvait déduire de la formule trouvée quelques unes des lois fondamentales de l'Electrostatique.

Repasser, au moyen des formules (1) et (2), de l'expression de la variation infiniment petite d'un potentiel électrostatique à l'expression de la variation du potentiel d'un système polarisé sur lui même, c'est l'objet du Chapître III.

Au Chapître IV, nous faisons usage des résultats obtenus pour traiter le problème de l'équilibre d'un fluide incompressible doué de force coercitive; dans nos *Leçons sur l'Electricité et la Magnétisme*, nous avions déjà obtenu les conditions de cet équilibre; mais l'une de ces conditions était faussée par l'omission du terme complémentaire introduit par M. Liénard, et la démonstration des autres laissait à désirer au point de vue de la rigueur.

Enfin, au Chapître V, nous établissons les conditions générales de l'équilibre d'un fluide compressible dénué de force coercitive.

Dans les deux premiers Chapîtres de ce Mémoire, nous avons évité d'examiner le cas où la surface de contact de deux corps porte une *couche électrique double;* l'étude des couches électriques doubles présente des difficultés spéciales que nous examinerons dans un travail spécial.

Dans les deux derniers Chapîtres, nous avons borné notre exposé aux fluides polarisés; l'équilibre des solides élastiques polarisés se traitera sans peine en suivant les méthodes indiquées dans nos *Leçons sur l'Electricité et le Magnétisme* et en corrigeant, au moyen des calculs donnés ici, la forme des conditions aux limites.

Nous n'avons pas repris, non plus, l'étude de l'influence que le magnétisme exerce sur une dissolution d'un sel magnétique dans un liquide non magnétique; le lecteur trouvera sans peine de quelle manière les conditions aux limites doivent être corrigées par l'introduction du terme complémentaire de M. Liénard; dans le cas, seul réalisable pratiquement, où le sel est peu magnétique, ce terme est négligeable ; dans ce cas, les résultats que nous avons donnés autrefois deviennent tous exacts.

CHAPÎTRE I.

Variation du Potentiel électrostatique d'un Système.

Considérons un système électrisé portant à la fois une distribution électrique solide à l'intérieur des masses continues qui le forment et une distribution superficielle sur les surfaces de discontinuité qui limitent ces masses ; nous laisserons de côté, dans le présent travail, le cas où ces surfaces porteraient une *couche électrique double ;* nous nous proposons de consacrer à ce cas un mémoire spécial.

Soient : M_1 un point situé à l'intérieur de l'une des masses continues qui forment le système :

dv_1, un élément de volume entourant le point M_1 ;
ρ_1, la densité électrique solide au point M_1 ;
μ_1, un point situé sur une surface de discontinuité ;
dS_1, une aire élémentaire découpée sur cette surface, autour du point μ_1 ;
σ_1, la densité électrique superficielle au point μ_1.

Au point $M(x, y, z)$, la distribution électrique solide aura pour fonction potentielle,

$$U(M) = U(x, y, z) = \int \frac{\rho_1}{r_1} dv_1, \tag{1}$$

r_1 étant la distance $\overline{MM_1}$ et l'intégrale s'étendant à tout espace rempli par une masse continue électrisée.

Au point $M(x, y, z)$, la distribution électrique superficielle aura pour fonction potentielle,

$$W(M) = W(x, y, z) = \mathbf{S} \frac{\sigma_1}{r_1} dS_1, \tag{2}$$

r_1 désignant la distance $\overline{M\mu_1}$ et l'intégrale s'étendant à toutes les surfaces de discontinuité électrisées.

La fonction potentielle totale aura pour valeur, au point $M(x, y, z)$,

$$\begin{aligned} V(x, y, z) = V(M) &= U(M) + W(M) \\ &= \int \frac{\rho_1}{r_1} dv_1 + \mathbf{S} \frac{\sigma_1}{r_1} dS_1. \end{aligned} \tag{3}$$

Le Potentiel électrostatique du système a pour valeur,

$$Y = \frac{\varepsilon}{2}\int \rho\, U dv + \varepsilon \mathbf{S} \sigma\, U dS + \frac{\varepsilon}{2} \mathbf{S} \sigma\, W dS, \tag{4}$$

ε étant la constante fondamentale de l'électrostatique.

Il est la somme de trois termes:

1°. Le Potentiel de la distribution solide sur elle même: $\frac{\varepsilon}{2}\int \rho\, U dv$;

2°. Le Potentiel de la distribution solide sur la distribution superficielle: $\varepsilon \mathbf{S} \sigma\, U dS$;

3°. Le Potentiel de la distribution superficielle sur elle même: $\frac{\varepsilon}{2} \mathbf{S} \sigma\, W dS$..

Prenons maintenant deux états du système infiniment voisins l'un de l'autre.

Entre chaque point *géométrique* M du système dans le premier état et chaque point *géométrique* M' dans le second état, établissons une correspondance univoque assujettie aux conditions suivantes:

1°. Deux points correspondants M, M', sont toujours infiniment voisins l'un de l'autre.

2°. A tout volume v du premier système, tout entier compris à l'intérieur d'une même masse continue, correspond un volume v' du second système, tout entier compris à l'intérieur d'une même masse continue, et réciproquement.

3°. A toute aire S du premier système, tout entière tracée sur une même surface de discontinuité, correspond une aire S' du second système, tout entière tracée sur une même surface de discontinuité, et réciproquement.

(Ces deux conditions excluent la possibilité de toute scission, de toute déchirure, durant la déformation.)

4°. En tout point d'un volume tel que v, la déformation fait naître des dilatations et des glissements qui sont infiniment petits.

5°. En tout point d'une aire telle que S, qui se transforme en sa correspondante S', les déformations sont infiniment petites.

6°. Les densités électriques solides ρ, ρ', en deux points correspondants M, M', diffèrent infiniment peu l'une de l'autre.

7°. Les densités électriques superficielles σ, σ', en deux points correspondants μ, μ', diffèrent infiniment peu l'une de l'autre.*

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* Dans un grand nombre de cas, les hypothèses précédentes seront vérifiées, si l'on fait correspondre entre eux les deux points *géométriques* M, M', positions initiale et finale d'un même point *matériel;* on pourra alors, si on le juge utile, adopter ce mode de correspondance, mais on ne sera jamais tenu de la faire; on pourra toujours, si l'on y trouve avantage, en établir un autre.

Soient Y la valeur du potentiel électrostatique du système dans le premier état et Y' la valeur du potentiel électrostatique du système dans le second état. Nous allons chercher à calculer l'infiniment petit principal de la différence $(Y' - Y)$, infiniment petit principal que nous désignerons par δY.

1°. *Calcul du terme principal de* $[U'(M') - U(M)]$.

En un point M du système, pris dans le premier état, la distribution électrique solide que porte le système dans cet état admet une fonction potentielle $U(M)$; au point correspondant M' du système pris dans le second état, la distribution électrique solide que porte le système dans cet état admet une fonction potentielle $U'(M')$. *La différence* $[U'(M') - U(M)]$ *est infiniment petite;* cette proposition est évidente lorsque le point M et, par conséquent, le point M', sont extérieurs aux masses électrisées; une démonstration est nécessaire dans le cas où le point M, et, partant, le point M', appartiennent à une masse électrisée.

Soit P la limite supérieure des valeurs absolues que peut prendre la densité électrique solide ρ en un point quelconque du système et en l'un quelconque des états compris dans l'ensemble d'états que l'on veut considérer.

Soit m un point quelconque du système en l'un de ses états. Du point m comme centre, décrivons une sphère Σ de rayon R; la fonction potentielle au point m de la distribution électrique solide répandue à l'intérieur de cette sphère sera inférieure, en valeur absolue, à $2\pi PR^2$. On peut donc prendre R assez petit pour que cette fonction potentielle soit inférieure en valeur absolue à une quantité donnée d'avance η. R étant ainsi déterminé, prenons le second état du système assez voisin du premier pour que $\overline{MM'}$ soit inférieur à R. Posons:

$$U(M) = u(M) + \mathfrak{U}(M),$$

$u(M)$ étant, au point M, la fonction potentielle de la distribution solide que renferme, dans le premier état du système, une sphère de rayon R ayant le point M pour centre et $\mathfrak{U}(M)$ étant la fonction potentielle de la distribution solide qui demeure extérieure à cette sphère. Posons de même

$$U'(M') = u'(M') + \mathfrak{U}'(M'),$$

$u'(M')$ étant, au point M', la fonction potentielle de la distribution solide que renferme, dans le second état du système, une sphère de rayon R ayant le point M' pour centre et $\mathfrak{U}'(M')$ étant la fonction potentielle de la distribution superficielle qui demeure extérieure à cette sphère. Nous aurons sûrement

$$|u(M)| < \eta, \quad |u'(M')| < \eta.$$

D'autre part, le point M est intérieur à la sphère de rayon R ayant le point M' pour centre et le point M' est intérieur à la sphère de rayon R ayant le point M pour centre ; il est alors évident que $\mathfrak{U}'(M')$ tend d'une manière continue vers $\mathfrak{U}(M)$, lorsque le second état du système tend vers le premier ; on peut par conséquent prendre le second état assez voisin du premier pour que l'on ait

$$|\mathfrak{U}'(M') - \mathfrak{U}(M)| < \eta.$$

On aura, alors

$$|U'(M') - U(M)| < 3\eta.$$

On peut donc prendre le second état du système assez voisin du premier pour que la différence $[U'(M') - U(M)]$ soit inférieure en valeur absolue à une quantité donnée d'avance ; cette différence est donc infiniment petite, comme nous l'avions annoncé.

Nous sommes assurés que la différence $[U'(M') - U(M)]$ peut être regardée comme un infiniment petit du premier ordre ; nous allons maintenant évaluer le terme principal de cette quantité.

Nous avons

$$U'(M') - U(M) = \int \frac{\rho'_1}{M'M'_1}\, dv'_1 - \int \frac{\rho_1}{MM_1}\, dv_1,$$

la seconde intégrale s'étendant à tous les éléments dv_1 du premier état, et la première à tous les éléments *correspondants* dv'_1 du second état.

Nous pouvons exprimer cette différence au moyen d'intégrales toutes étendues aux éléments dv_1 du premier état, et écrire :

$$\begin{aligned} U'(M') - U(M) = & \int \frac{\rho'_1 - \rho_1}{MM_1}\, dv_1 \\ & + \int \rho_1 \left(\frac{1}{M'M'_1} - \frac{1}{MM_1} \right) dv_1 \\ & + \int \frac{\rho_1}{MM_1} \, \frac{dv'_1 - dv_1}{dv_1}\, dv_1 + K, \end{aligned} \tag{5}$$

avec

$$\begin{aligned} K = & \int \left(\frac{\rho'_1}{M'M'_1} - \frac{\rho_1}{MM_1} \right) \frac{dv'_1 - dv_1}{dv_1}\, dv_1 \\ & + \int (\rho'_1 - \rho_1) \left(\frac{1}{M'M'_1} - \frac{1}{MM_1} \right) dv_1. \end{aligned} \tag{6}$$

Posons

$$\lambda_1 = \lambda(M_1) = \frac{dv'_1}{dv_1}; \tag{7}$$

$\lambda(M_1)$ sera, d'après nos conventions, une fonction du point M_1, qui gardera, pour tous les points M_1, une valeur infiniment voisine de 1.

Nos conventions nous permettent également d'écrire

$$\frac{dv_1' - dv_1}{dv_1} = \theta\varpi(M_1) = \theta\varpi_1, \tag{8}$$

$$\rho_1' - \rho_1 = \vartheta p(M_1) = \vartheta p_1, \tag{9}$$

θ et ϑ étant deux facteurs infiniment petits indépendants du point M_1, tandis que $\varpi(M_1)$ et $p(M_1)$ sont deux fonctions finies du point M_1.

Nous pouvons alors écrire l'égalité (6) sous la forme

$$\begin{aligned} K = \theta\left(\int \frac{\lambda_1\varpi_1\rho_1'}{\overline{M'M_1'}}\,dv_1' - \int \frac{\varpi_1\rho_1}{\overline{MM_1}}\,dv_1\right) \\ + \vartheta\left(\int \frac{\lambda_1 p_1}{\overline{M'M_1'}}\,dv_1' - \int \frac{p_1}{\overline{MM_1}}\,dv_1\right). \end{aligned} \tag{10}$$

La différence

$$\int \frac{\lambda_1\varpi_1\rho_1'}{\overline{M'M_1'}}\,dv_1' - \int \frac{\varpi_1\rho_1}{\overline{MM_1}}\,dv_1$$

est ce que deviendrait la différence $[U'(M') - U(M)]$ si, au point M_1 du système pris dans le premier état, la densité électrique solide avait la valeur finie $\varpi_1\rho_1$ et si, au point correspondant M_1' du système pris dans le second état, la densité électrique solide avait la valeur, infiniment voisine de la précédente, $\lambda_1\varpi_1\rho_1$; dès lors, nous savons que cette différence est infiniment petite.

La différence

$$\int \frac{\lambda_1 p_1}{\overline{M'M_1'}}\,dv_1' - \int \frac{p_1}{\overline{MM_1}}\,dv_1$$

est ce que deviendrait la différence $[U'(M') - U(M)]$, si, au point M_1 du système pris dans le premier état, la densité électrique solide avait la valeur finie p_1 et si, au point correspondant M_1' du système pris dans le second état, la densité électrique solide avait la valeur, infiniment voisine de la précédente, $\lambda_1 p_1$; dès lors, nous savons que cette différence est infiniment petite.

Comme ϑ et θ sont deux facteurs infiniment petits, l'égalité (10) nous apprend que *K est un infiniment petit d'ordre supérieur au premier.*

Le terme principal de $[U'(M') - U(M)]$ est donc formé par l'ensemble des termes explicitement écrits au second membre de l'égalité (5).

2°. *Variation du potentiel de la distribution électrique solide sur elle-même.*

Nous aurons

$$\begin{aligned}\delta\int U\rho dv &= \int U'\rho' dv' - \int U\rho dv \\ &= \int (U' - U)\rho dv \\ &\quad + \int (\rho' - \rho)\, U dv \\ &\quad + \int \rho U \frac{dv' - dv}{dv} dv + H, \end{aligned} \tag{11}$$

avec

$$H = \int (U' - U)(\rho' dv' - \rho dv) + \int U(\rho' - \rho)\frac{dv' - dv}{dv} dv. \tag{12}$$

En vertu des égalités (8) et (9), nous pourrons écrire

$$\begin{aligned} H = \theta \int (U' - U)\, \varpi\rho dv &+ \vartheta \int (U' - U)\, y dv \\ &+ \theta\vartheta \int U' p\varpi dv. \end{aligned} \tag{13}$$

La différence

$$U' - U = U'(M) - U(M)$$

étant infiniment petite, ainsi que les deux facteurs θ et ϑ, cette égalité (13) montre que *tous les termes qui composent la quantité H sont infiniment petits d'ordre supérieur au premier.* Le terme principal de $\delta\int U\rho dv$ est donc représenté par l'ensemble des termes explicitement écrits au second membre de l'égalité (11). On peut d'ailleurs, dans le premier de ces termes, remplacer la différence $(U' - U)$ par son infiniment petit principal, c'est-à-dire par l'ensemble des termes explicitement écrits au second membre de l'égalité (5). On obtient ainsi l'égalité suivante :

$$\begin{aligned} \delta\int U\rho dv = &\int (\rho' - \rho)\, U dv \\ &+ \int\int \frac{\rho_1' - \rho_1}{\overline{MM_1}} \rho dv_1\, dv \\ &+ \int\int \rho_1 \left(\frac{1}{\overline{M'M_1'}} - \frac{1}{\overline{MM_1}}\right) \rho dv_1\, dv \\ &+ \int\int \frac{\rho_1}{\overline{MM_1}}\, \frac{dv_1' - dv_1}{dv_1} \rho dv_1\, dv \\ &+ \int \rho U \frac{dv' - dv}{dv} dv. \end{aligned} \tag{14}$$

Transformons cette égalité (14).

Plaçons, au point M_1, une densité électrique solide $R_1 = \rho_1' - \rho_1$; soit

$$\mathfrak{u}(M) = \int \frac{R_1}{\overline{MM_1}} dv_1$$

la valeur, au point M, de la distribution ainsi obtenue. Une identité bien connue, due à Gauss, nous donnera

$$\int \mathfrak{u} \rho dv = \int U R dv,$$

ou bien

$$\int\int \frac{\rho_1' - \rho_1}{\overline{MM_1}} \rho dv_1 dv = \int (\rho' - \rho) U dv. \tag{15}$$

Nous aurons identiquement

$$\begin{aligned} \int\int \frac{\rho_1}{\overline{MM_1}} \frac{dv_1' - dv_1}{dv_1} \rho dv_1 dv &= \int\int \frac{\rho}{\overline{M_1 M}} \frac{dv_1' - dv_1}{dv_1} \rho_1 dv dv_1 \\ &= \int \rho_1 U_1 \frac{dv_1' - dv_1}{dv_1} dv_1 \\ &= \int \rho U \frac{dv' - dv}{dv} dv. \end{aligned} \tag{16}$$

Enfin nous pouvons écrire

$$\begin{aligned} \int\int \rho\rho_1 \left(\frac{1}{\overline{M'M_1'}} - \frac{1}{\overline{MM_1}} \right) dv dv_1 &= - \int\int \frac{\overline{M'M_1'}^2 - \overline{MM_1}^2}{\overline{MM_1} . \overline{M'M_1'} (\overline{MM_1} + \overline{M'M_1'})} \rho\rho_1 dv dv_1 \\ = - \int\int \frac{(x_1'-x')^2+(y_1'-y')^2+(z_1'-z')^2-(x_1-x)^2-(y_1-y)^2-(z_1-z)^2}{\overline{MM_1} . \overline{M'M_1'} (\overline{MM_1} + \overline{M'M_1'})} & \rho\rho_1 dv dv_1 . \end{aligned} \tag{17}$$

Nous avons

$$\begin{aligned} & \int\int \frac{(x_1' - x')^2 - (x_1 - x)^2}{\overline{MM_1} . \overline{M'M_1'} (\overline{MM_1} + \overline{M'M_1'})} \rho\rho_1 dv dv_1 \\ = & \int\int \frac{[(x_1' - x_1) - (x' - x)] (x_1' - x' + x_1 - x)}{\overline{MM_1} . \overline{M'M_1'} (\overline{MM_1} + \overline{M'M_1'})} \rho\rho_1 dv dv_1 \\ = & \int \rho_1 (x_1' - x_1) \left[\int \frac{x_1' - x' + x_1 - x}{\overline{MM_1} . \overline{M'M_1'} (\overline{MM_1} + \overline{M'M_1'})} \rho dv \right] dv_1 \\ + & \int \rho (x' - x) \left[\int \frac{x' - x_1' + x - x_1}{\overline{M_1 M} . \overline{M_1' M'} (\overline{M_1 M} + \overline{M_1' M'})} \rho_1 dv_1 \right] dv \\ = & 2 \int \rho (x' - x) \left[\int \frac{x' - x_1' + x - x_1}{\overline{MM_1} . \overline{M'M_1'} + (\overline{MM_1} + \overline{M'M_1'})} \rho_1 dv_1 \right] dv. \end{aligned} \tag{18}$$

Mais il résulte des hypothèses faites que les rapports $\frac{x' - x_1'}{x - x_1}$, $\frac{\overline{M'M_1'}}{\overline{MM_1}}$, diffèrent infiniment peu de 1, même lorsque $(x - x_1)$ et $\overline{MM_1}$ sont infiniment petits; on peut donc écrire

$$\begin{aligned}&\int \frac{x' - x'_1 + x - x_1}{\overline{MM_1}.\overline{M'M'_1}\,(\overline{MM_1} + \overline{M'M'_1})}\,\rho_1 dv_1 \\ &= \int \frac{x - x_1}{\overline{MM_1}^3}\,\rho_1 dv_1 + \int K \frac{x - x_1}{\overline{MM_1}^3}\,\rho_1 dv_1,\end{aligned} \tag{19}$$

K étant infiniment petit. Au second membre de cette égalité (19), le premier terme est évidemment fini et le second infiniment petit. Cette égalité, jointe à l'égalité (18), montre alors qu'en négligeant les infiniment petits d'ordre supérieur au premier, on peut écrire la première des égalités

$$\left.\begin{aligned}&\int\int \frac{(x'_1 - x')^2 - (x_1 - x)^2}{\overline{MM_1}.\overline{M'M'_1}\,(\overline{MM_1} + \overline{M'M'_1})}\,\rho\rho_1 dv\,dv_1 = 2\int \rho\,(x' - x)\left(\int \frac{x - x_1}{\overline{MM_1}^3}\,\rho_1 dv_1\right) dv, \\ &\int\int \frac{(y'_1 - y')^2 - (y_1 - y)^2}{\overline{MM_1}.\overline{M'M'_1}\,(\overline{MM_1} + \overline{M'M'_1})}\,\rho\rho_1 dv\,dv_1 = 2\int \rho\,(y' - y)\left(\int \frac{y - y_1}{\overline{MM_1}^3}\,\rho_1 dv_1\right) dv, \\ &\int\int \frac{(z'_1 - z')^2 - (z_1 - z)^2}{\overline{MM_1}.\overline{M'M'_1}\,(\overline{MM_1} + \overline{M'M'_1})}\,\rho\rho_1 dv\,dv_1 = 2\int \rho\,(z' - z)\left(\int \frac{z - z_1}{\overline{MM_1}^3}\,\rho_1 dv_1\right) dv;\end{aligned}\right\} \tag{20}$$

les deux autres s'établisent d'une manière analogue.

Posons

$$\left.\begin{aligned}\xi(x, y, z) &= \xi(M) = \varepsilon \int \frac{x - x_1}{\overline{MM_1}^3}\,\rho_1 dv_1, \\ \eta(x, y, z) &= \eta(M) = \varepsilon \int \frac{y - y_1}{\overline{MM_1}^3}\,\rho_1 dv_1, \\ \zeta(x, y, z) &= \zeta(M) = \varepsilon \int \frac{z - z_1}{\overline{MM_1}^3}\,\rho_1 dv_1\end{aligned}\right\} \tag{21}$$

et les égalités (17) et (20) nous permettront d'écrire, en négligeant les infiniment petits d'ordre supérieur,

$$\varepsilon \int\int \rho\rho_1 \left(\frac{1}{\overline{M'M'_1}} - \frac{1}{\overline{MM_1}}\right) dv\,dv_1 = -2\int \rho\,[\xi(x' - x) + \eta(y' - y) + \zeta(z' - z)]\,dv. \tag{22}$$

Les égalités (14), (15), (16) et (22) donnent, en dernière analyse,

$$\begin{aligned}\frac{\varepsilon}{2}\,\delta \int U\rho dv &= \varepsilon \int (\rho' - \rho)\,U dv \\ &+ \varepsilon \int \rho U \frac{dv' - dv}{dv}\,dv \\ &- \int \rho\,[\xi(x' - x) + \eta(y' - y) + \zeta(z' - z)]\,dv.\end{aligned} \tag{23}$$

Nous avons eu soin, au cours des raisonnements qu'on vient de lire, d'invoquer seulement des propriétés communes aux distributions électriques solides et aux distributions électriques superficielles; nous pourrons alors nous abstenir de développer de nouveau ces raisonnements dans les deux cas qui nous restent à traiter.

3°. *Variation du potentiel de la distribution solide sur la distribution superficielle.*

En posant

$$\left.\begin{aligned} \mathfrak{X}(M) &= \varepsilon \mathop{\mathrm{S}} \frac{x - x_1}{\overline{M\mu_1}^3} \sigma_1 dS_1, \\ \mathfrak{Y}(M) &= \varepsilon \mathop{\mathrm{S}} \frac{y - y_1}{\overline{M\mu_1}^3} \sigma_1 dS_1, \\ \mathfrak{Z}(M) &= \varepsilon \mathop{\mathrm{S}} \frac{z - z_1}{\overline{M\mu_1}^3} \sigma_1 dS_1, \end{aligned}\right\} \tag{24}$$

nous trouverons pour expression de la variation de la distribution solide sur la distribution superficielle:

$$\begin{aligned} \varepsilon\delta \mathop{\mathrm{S}} \sigma U dS = {} & \varepsilon \mathop{\mathrm{S}} (\sigma' - \sigma) U dS \\ & + \varepsilon \int (\rho' - \rho) W dv \\ & + \varepsilon \mathop{\mathrm{S}} U\sigma \frac{dS' - dS}{dS} dS \\ & + \varepsilon \int W\rho \frac{dv' - dv}{dv} dv \\ & - \mathop{\mathrm{S}} \sigma [\xi(x' - x) + \eta(y' - y) + \zeta(z' - z)] dS \\ & - \int \rho [\mathfrak{X}(x' - x) + \mathfrak{Y}(y' - y) + \mathfrak{Z}(z' - z)] dv. \end{aligned} \tag{25}$$

4°. *Variation du potentiel de la distribution superficielle sur elle même.*

Nous aurons de même

$$\begin{aligned} \frac{\varepsilon}{2} \delta \mathop{\mathrm{S}} \sigma W dS = {} & \varepsilon \mathop{\mathrm{S}} (\sigma' - \sigma) W dS \\ & + \varepsilon \mathop{\mathrm{S}} W\sigma \frac{dS' - dS}{dS} dS \\ & - \mathop{\mathrm{S}} \sigma [\mathfrak{X}(x' - x) + \mathfrak{Y}(y' - y) + \mathfrak{Z}(z' - z)] dS. \end{aligned} \tag{26}$$

5°. *Variation du potentiel électrostatique d'un système.*

Posons

$$\left.\begin{aligned} X(M) &= \xi(M) + \mathfrak{X}(M) = \varepsilon\int \frac{x - x_1}{\overline{MM_1}^3}\rho_1 dv_1 + \varepsilon\,\mathbf{S}\frac{x - x_1}{\overline{M\mu_1}^3}\sigma_1 dS_1, \\ Y(M) &= \eta(M) + \mathfrak{Y}(M) = \varepsilon\int \frac{y - y_1}{\overline{MM_1}^3}\rho_1 dv_1 + \varepsilon\,\mathbf{S}\frac{y - y_1}{\overline{M\mu_1}^3}\sigma_1 dS_1, \\ Z(M) &= \zeta(M) + \mathfrak{Z}(M) = \varepsilon\int \frac{z - z_1}{\overline{MM_1}^3}\rho_1 dv_1 + \varepsilon\,\mathbf{S}\frac{z - z_1}{\overline{M\mu_1}^3}\sigma_1 dS_1. \end{aligned}\right\} \quad (27)$$

Posons, en outre,

$$\rho' - \rho = \delta\rho, \quad x' - x = \delta x, \quad y' - y = \delta y, \quad z' - z = \delta z.$$

Une formule connue nous donnera

$$\frac{dv' - dv}{dv} = \frac{\partial\delta x}{\partial x} + \frac{\partial\delta y}{\partial y} + \frac{\partial\delta z}{\partial z}.$$

Les égalitées (4), (23), (25) et (26) nous donneront l'expression suivante pour la variation du potentiel électrostatique d'un système

$$\begin{aligned} \delta Y = {} & \varepsilon\int V\delta\rho\, dv + \varepsilon\,\mathbf{S}\, V\delta\sigma\, dS \\ & - \int \rho\,(X\delta x + Y\delta y + Z\delta z)\, dv - \mathbf{S}\,\sigma\,(X\delta x + Y\delta y + Z\delta z)\, dS \\ & + \varepsilon\int V\rho\left(\frac{\partial\delta x}{\partial x} + \frac{\partial\delta y}{\partial y} + \frac{\partial\delta z}{\partial z}\right) dv + \varepsilon\,\mathbf{S}\, V\sigma\frac{dS' - dS}{dS}\, dS. \end{aligned} \quad (28)$$

Dans certains cas, il y a avantage à transformer cette égalité au moyen de relations que nous allons écrire.

En tout point *intérieur* à une masse continue électrisée, on a

$$X = -\varepsilon\frac{\partial V}{\partial x}, \quad Y = -\varepsilon\frac{\partial V}{\partial y}, \quad Z = -\varepsilon\frac{\partial V}{\partial z}. \quad (29)$$

Ces égalités perdent tout sens en un point pris sur une surface de discontinuité électrisée.

Soient 1 et 2 les deux régions de l'espace qui sont situées de part et d'autre de la surface S; soient N_1 la demi-normale menée vers l'intérieur de la région 1 et N_2 la demi-normale menée vers l'intérieur de la région 2. La surface S est une surface de discontinuité pour les dérivées partielles du premier ordre de la fonction potentielle. Si M est le point de la surface S auquel se rapportent les quantités X, Y, Z; si $M_1(x_1, y_1, z_1)$ et $M_2(x_2, y_2, z_2)$ sont deux points infiniment voisins du point M et situés le premier dans la région 1, le second dans la région 2, on a

$$\left.\begin{aligned} X &= -\varepsilon\left[\frac{\partial V}{\partial x_1} + 2\pi\sigma\cos(N_1, x)\right] = -\varepsilon\left[\frac{\partial V}{\partial x_2} + 2\pi\sigma\cos(N_2, x)\right], \\ Y &= -\varepsilon\left[\frac{\partial V}{\partial y_1} + 2\pi\sigma\cos(N_1, y)\right] = -\varepsilon\left[\frac{\partial V}{\partial y_2} + 2\pi\sigma\cos(N_2, y)\right], \\ Z &= -\varepsilon\left[\frac{\partial V}{\partial z_1} + 2\pi\sigma\cos(N_1, z)\right] = -\varepsilon\left[\frac{\partial V}{\partial z_2} + 2\pi\sigma\cos(N_2, z)\right]. \end{aligned}\right\} \qquad (30)$$

Les identités

$$\begin{aligned} \cos(N_1, x) + \cos(N_2, x) &= 0, \\ \cos(N_1, y) + \cos(N_2, y) &= 0, \\ \cos(N_1, z) + \cos(N_2, z) &= 0, \end{aligned}$$

permettent de transformer les égalités (30) en

$$\left.\begin{aligned} X &= -\frac{\varepsilon}{2}\left(\frac{\partial V}{\partial x_1} + \frac{\partial V}{\partial x_2}\right), \\ Y &= -\frac{\varepsilon}{2}\left(\frac{\partial V}{\partial y_1} + \frac{\partial V}{\partial y_2}\right), \\ Z &= -\frac{\varepsilon}{2}\left(\frac{\partial V}{\partial z_1} + \frac{\partial V}{\partial z_2}\right). \end{aligned}\right\} \qquad (31)$$

De ces égalités (30) et (31) nous pouvons encore déduire les égalités

$$\left.\begin{aligned} \frac{\partial V}{\partial x_1} - \frac{\partial V}{\partial x_2} &= -4\pi\sigma\cos(N_1, x), \\ \frac{\partial V}{\partial y_1} - \frac{\partial V}{\partial y_2} &= -4\pi\sigma\cos(N_1, y), \\ \frac{\partial V}{\partial z_1} - \frac{\partial V}{\partial z_2} &= -4\pi\sigma\cos(N_1, z). \end{aligned}\right\} \qquad (32)$$

Ces diverses formules nous seront utiles par la suite.

CHAPÎTRE II.

Application de la Formule précédente à quelques Questions d'Électrostatique.

§1.—*Conditions de l'équilibre électrique.*

Le potentiel thermodynamique interne d'un système électrisé a pour expression

$$F = F_0 + \int \Theta\rho dv + \mathop{\mathsf{S}} \Theta\rho dS + Y, \qquad (1)$$

Θ étant une quantité qui, sur un conducteur, varie d'un point à l'autre et F_0 la valeur que prendrait le potentiel thermodynamique interne du système ramené à l'état neutre.

Comme nous ne voulons pas, dans le présent mémoire, introduire la considération de couches électriques doubles, nous nous limiterons au cas où Θ *varie d'une manière continue d'un point à l'autre de toute masse conductrice connexe.*

Supposons que le conducteur se compose de deux masses conductrices séparément connexes, 1 et 2, limitées respectivement par les surfaces S_1, S_2. Imposons à ce système une variation de distribution électrique, sans changement de position ni de forme des corps conducteurs qui le constituent. Faisons correspondre les deux points *géométriques* M, M', où un même point *matériel* se trouve au début et à la fin de la modification ; les deux points M et M' coïncideront. En vertu de l'égalité (28) du Chapître I et de l'égalité (1), nous aurons

$$\begin{aligned} \delta F = \int (\varepsilon V + \Theta)\, \delta\rho_1\, dv_1 + \mathop{\mathrm{S}} (\varepsilon V + \Theta)\, \delta\sigma_1\, dS_1 \\ + \int (\varepsilon V + \Theta)\, \delta\rho_2\, dv_2 + \mathop{\mathrm{S}} (\varepsilon V + \Theta)\, \delta\sigma_2\, dS_2 . \end{aligned}$$

Les conditions d'équilibre s'obtiendront en exprimant que δF est égal à 0 pour toutes les variations virtuelles de la distribution électrique.

Si l'on désigne par Q_1 et Q_2 les charges *invariables* des corps 1 et 2, on aura

$$\int \rho_1\, dv_1 + \mathop{\mathrm{S}} \sigma_1\, dS_1 = Q_1,$$

$$\int \rho_2\, dv_2 + \mathop{\mathrm{S}} \sigma_2\, dS_2 = Q_2,$$

en sorte que les variations virtuelles de la distribution électrique seront assujetties aux conditions

$$\int \delta\rho_1\, dv_1 + \mathop{\mathrm{S}} \delta\sigma_1\, dS_1 = 0,$$

$$\int \delta\rho_2\, dv_2 + \mathop{\mathrm{S}} \delta\sigma_2\, dS_2 = 0.$$

C'est seulement lorsque ces conditions sont remplies que δF doit être égal à 0. Il doit donc exister deux constantes C_1, C_2, telles que l'on ait *identiquement*

$$\begin{aligned} \int (\varepsilon V + \Theta + C_1)\, \delta\rho_1\, dv_1 + \mathop{\mathrm{S}} (\varepsilon V + \Theta + C_1)\, \delta\sigma_1\, dS_1 \\ + \int (\varepsilon V + \Theta + C_2)\, \delta\rho_2\, dv_2 + \mathop{\mathrm{S}} (\varepsilon V + \Theta + C_2)\, \delta\sigma_2\, dS_2 = 0. \end{aligned}$$

Pour que cette identité ait lieu, il faut et il suffit que l'on ait:

1°. En tout point du corps 1 ou de la surface qui le limite,

$$\varepsilon V + \Theta + C_1 = 0. \tag{2}$$

2°. En tout point du corps 2 ou de la surface qui le limite,

$$\varepsilon V + \Theta + C_2 = 0. \tag{2 bis}$$

On obtient ainsi, par une méthode entièrement rigoureuse, les équations connues de l'équilibre électrique.

§ 2.—*Actions qui s'exercent entre des corps dont la forme et l'électrisation demeurent invariables.*

Imaginons un système formés de corps qui se déplacent de manière que chacun d'eux garde une figure invariable et que chaque point matériel entraîne la charge qu'il porte; tel est un système formé de solides parfaitement rigides et parfaitement mauvais conducteurs.

Prenons deux états voisins du système; à chaque point géométrique M du premier état faisons correspondre un point géométrique M' du second état, de telle sorte qu'un même point matériel se trouve en M au début de la modification et en M' à la fin; δx, δy, δz, sont alors, en chaque point géométrique, les composantes du déplacement subi par le point matériel qui se trouvait en ce point géométrique.

Les hypothèses faites nous donnent les égalités

$$\delta\rho = 0, \quad \delta\sigma = 0, \quad dv' - dv = 0, \quad dS' - dS = 0.$$

L'égalité (28) du Chapître I et l'égalité (1) donnent

$$\delta F = \delta F_0 - \int \rho (X\delta x + Y\delta y + Z\delta z)\, dv$$
$$- \mathop{\mathrm{S}} \sigma (X\delta x + Y\delta y + Z\delta z)\, dS.$$

Cette égalité nous montre que chaque élément matériel est soumis:

1°. Aux forces auxquelles il resterait soumis si le système était ramené à l'état neutre.

2°. A une force qui a pour composantes Xq, Yq, Zq, si l'on désigne par q la charge électrique totale de l'élément matériel considéré.

On retrouve ainsi, sous leur forme la plus générale, les lois de Coulomb, point de départ de l'électrostatique.

§3.—*Actions qui s'exercent sur des corps conducteurs électrisés.*

Nous supposerons, comme au §1, que d'un point à l'autre de toute masse conductrice connexe la quantité Θ varie d'une manière continue; de plus, pour simplifier nos recherches, nous supposerons que chaque masse conductrice demeure homogène même au voisinage des surfaces terminales; nous admettrons que Θ garde la même valeur en tout point de la masse conductrice, même au voisinage des surfaces terminales; nous regarderons cette valeur de Θ comme fonction de la seule densité D de la matière conductrice, *et nous supposerons cette densité invariable.*

Pour éviter les formules trop longues, nous réduirons le système à un conducteur unique.

Enfin nous établirons la correspondance entre les points M, M', en suivant la règle qui nous a servi au § précédent.

Nous trouverons sans peine l'égalité

$$\begin{aligned}\delta\int\Theta\rho\,dv+\delta\mathbf{S}\,\Theta\sigma\,dS&=\int\Theta\delta\rho\,dv+\mathbf{S}\,\Theta\delta\sigma\,dS\\&+\int\Theta\rho\,\frac{dv'-dv}{dv}\,dv+\mathbf{S}\,\Theta\sigma\,\frac{dS'-dS}{dS}\,dS.\end{aligned}\tag{3}$$

D'autre part, les égalités (28), (29) et (30) du Chapître I donnent l'égalité

$$\begin{aligned}\delta Y=&\int\varepsilon V\delta\rho\,dv+\mathbf{S}\,\varepsilon V\delta\sigma\,dS\\&+\int\varepsilon V\rho\,\frac{dv'-dv}{dv}\,dv+\mathbf{S}\,\varepsilon V\sigma\,\frac{dS'-dS}{dS}\,dS\\&+\int\varepsilon\rho\left(\frac{\partial V}{\partial x}\delta x+\frac{\partial V}{\partial y}\delta y+\frac{\partial V}{\partial z}\delta z\right)dv\\&+\mathbf{S}\,\varepsilon\sigma\left(\frac{\partial V}{\partial x_i}\delta x+\frac{\partial V}{\partial y_i}\delta y+\frac{\partial V}{\partial z_i}\delta z\right)dS\\&+\mathbf{S}\,2\pi\varepsilon\sigma^2\left[\cos(N_i,x)\,\delta x+\cos(N_i,y)\,\delta y+\cos(N_i,z)\,\delta z\right]dS,\end{aligned}\tag{4}$$

N_i désignant la demi-normale à la surface limite du conducteur, dirigée vers l'*intérieur* du conducteur, et (x_i, y_i, z_i) étant un point infiniment voisin de l'élément dS, mais situé à l'intérieur du conducteur.

Or l'égalité

$$\varepsilon V+\Theta+C=0,\tag{2}$$

permet d'écrire, en tout point intérieur au conducteur,

$$\frac{\partial V}{\partial x}=0, \quad \frac{\partial V}{\partial y}=0, \quad \frac{\partial V}{\partial z}=0. \tag{5}$$

Les égalités (1), (2), (3), (4) et (5) donnent alors l'égalité

$$\begin{aligned}\delta F = \delta F_0 - C\Big[\int\Big(\delta\rho + \rho\,\frac{dv'-dv}{dv}\Big)\,dv + \mathbf{S}\Big(\delta\sigma + \sigma\,\frac{dS'-dS}{dS}\Big)\,dS\Big]\\ + 2\pi\varepsilon\,\mathbf{S}\,\sigma^2\left[\cos(N_i,x)\,\delta x + \cos(N_i,y)\,\delta y + \cos(N_i,z)\,\delta z\right]dS.\end{aligned} \tag{6}$$

Mais si l'on désigne par Q la charge électrique *invariable* du conducteur, on aura

$$\int\rho\,dv + \mathbf{S}\,\sigma\,dS = Q$$

et, par conséquent,

$$\int\Big(\delta\rho + \rho\,\frac{dv'-dv}{dv}\Big)\,dv + \mathbf{S}\Big(\delta\sigma + \sigma\,\frac{dS'-dS}{dS}\Big)\,dS = 0. \tag{7}$$

Moyennant l'égalité (7), l'égalité (6) peut s'écrire

$$\delta F = \delta F_0 + 2\pi\varepsilon\,\mathbf{S}\,\sigma^2\left[\cos(N_i,x)\,\delta x + \cos(N_i,y)\,\delta y + \cos(N_i,z)\,\delta z\right]dS.$$

Cette égalité nous fournit la proposition suivante :

Les forces qui agissent sur un conducteur électrisé déformable, mais incompressible, se composent :

1°. *Des forces qui agiraient sur le conducteur ramené à l'état neutre.*

2°. *D'une force appliquée à chaque élément dS de la surface qui limite le conducteur ; cette force est normale à l'élément dS et dirigée vers l'extérieur du conducteur ; elle a pour valeur*

$$T = 2\pi\varepsilon\sigma^2\,dS.$$

Ce théorème est bien connu.

CHAPÎTRE III.

Variation du Potentiel d'un Système de Diélectriques polarisés.

Nous allons maintenant—et c'est l'objet principal de ce mémoire—faire usage de l'égalité fondamentale obtenue au Chapître I pour étudier les milieux magnétiques ou diélectriques ; afin d'éviter toute confusion, nous supposerons constamment dans notre analyse qu'il s'agisse de corps diélectriques ; le lecteur apercevra sans peine les légères modifications qu'il faudrait faire subir à notre exposé pour l'appliquer aux corps magnétiques; la principale modification consisterait à faire égale à 1 la constante ε des lois de Coulomb.

Soient A, B, C, les composantes de la polarisation en un point (x, y, z) intérieur à un diélectrique. Le Théorème fondamental et bien connu sur lequel nous nous appuierons est le suivant:

On peut sans changer ni la fonction potentielle, ni le potentiel du système, remplacer la polarisation diélectrique par une distribution électrique fictive définie de la manière suivante:

La distribution fictive a pour densité solide, en tout point intérieur à une masse diélectrique continue

$$\rho = -\left(\frac{\partial \mathrm{A}}{\partial x} + \frac{\partial \mathrm{B}}{\partial y} + \frac{\partial \mathrm{C}}{\partial z}\right); \tag{1}$$

elle a pour densité superficielle, en tout point de la surface terminale d'un diélectrique,

$$\sigma = -\left[\mathrm{A} \cos(N_i, x) + \mathrm{B} \cos(N_i, y) + \mathrm{C} \cos(N_i, z)\right], \tag{2}$$

N_i étant la demi-normale dirigée vers l'intérieur diélectrique; elle a pour densité superficielle, en tout point de la surface de contact de deux masses diélectriques 1 *et* 2,

$$\begin{aligned} \sigma_{12} = &-\left[\mathrm{A}_1 \cos(N_1, x) + \mathrm{B}_1 \cos(N_1, y) + \mathrm{C}_1 \cos(N_1, z)\right] \\ &-\left[\mathrm{A}_2 \cos(N_2, x) + \mathrm{B}_2 \cos(N_2, y) + \mathrm{C}_2 \cos(N_2, z)\right], \end{aligned} \tag{3}$$

N_1, N_2, étant les deux demi-normales dirigées respectivement vers l'intérieur de la masse 1 *et vers l'intérieur de la masse* 2.

Imaginons deux diélectriques 1 et 2, dont S_1, S_2 sont les surfaces libres et dont S_{12} est la surface de contact. Soit Y le potentiel de la polarisation diélectrique sur elle même. En vertu des égalités (1), (2) et (3) du présent chapître et des égalités (28), (29), (30) et (31) du Chapître I, nous pourrons écrire

$$\begin{aligned} -\delta Y = {} & \varepsilon \int_1 V \delta \left(\frac{\partial \mathrm{A}_1}{\partial x} + \frac{\partial \mathrm{B}_1}{\partial y} + \frac{\partial C_1}{\partial z}\right) dv_1 \\ & + \varepsilon \int_1 \left(\frac{\partial \mathrm{A}_1}{\partial x} + \frac{\partial \mathrm{B}_1}{\partial y} + \frac{\partial \mathrm{C}_1}{\partial z}\right)\left(\frac{\partial V}{\partial x}\delta x + \frac{\partial V}{\partial y}\delta y + \frac{\partial V}{\partial z}\delta z\right) dv_1 \\ & + \varepsilon \int_1 V \left(\frac{\partial \mathrm{A}_1}{\partial x} + \frac{\partial \mathrm{B}_1}{\partial y} + \frac{\partial \mathrm{C}_1}{\partial z}\right)\left(\frac{\partial \delta x}{\partial x} + \frac{\partial \delta y}{\partial y} + \frac{\partial \delta z}{\partial z}\right) dv_1 \\ & + \varepsilon \mathrm{S}_{S_1} V \delta \left[\mathrm{A}_1 \cos(N_i, x) + \mathrm{B}_1 \cos(N_i, y) + \mathrm{C}_1 \cos(N_i, z)\right] dS_1 \\ & + \varepsilon \mathrm{S}_{S_{12}} V \delta \left[\mathrm{A}_1 \cos(N_1, x) + \mathrm{B}_1 \cos(N_1, y) + \mathrm{C}_1 \cos(N_1, z)\right] dS_{12} \end{aligned}$$

$$\begin{aligned}
&+ \varepsilon \mathop{\mathrm{S}}_{S_1} [\mathrm{A}_1 \cos(N_i, x) + \mathrm{B}_1 \cos(N_i, y) + \mathrm{C}_1 \cos(N_i, z)] \left(\frac{\partial V}{\partial x_i}\delta x + \frac{\partial V}{\partial y_i}\delta y + \frac{\partial V}{\partial z_i}\delta z\right) dS_1 \\
&+ \varepsilon \mathop{\mathrm{S}}_{S_{12}} [\mathrm{A}_1 \cos(N_1, x) + \mathrm{B}_1 \cos(N_1, y) + \mathrm{C}_1 \cos(N_1, z)] \left(\frac{\partial V}{\partial x_1}\delta x + \frac{\partial V}{\partial y_1}\delta y + \frac{\partial V}{\partial z_1}\delta z\right) dS_{12} \\
&- 2\pi\varepsilon \mathop{\mathrm{S}}_{S_1} [\mathrm{A}_1 \cos(N_i, x) + \mathrm{B}_1 \cos(N_i, y) + \mathrm{C}_1 \cos(N_i, z)]^2 \\
&\qquad\qquad \times [\cos(N_i, x)\,\delta x + \cos(N_i, y)\,\delta y + \cos(N_i, z)\,\delta z]\, dS_1 \\
&+ \varepsilon \mathop{\mathrm{S}}_{S_1} V [\mathrm{A}_1 \cos(N_i, x) + \mathrm{B}_1 \cos(N_i, y) + \mathrm{C}_1 \cos(N_i, z)] \frac{dS_1' - dS_1}{dS_1} dS_1 \\
&+ \text{etc.}, \\
&- \frac{\varepsilon}{2} \mathop{\mathrm{S}}_{S_{12}} [\mathrm{A}_1 \cos(N_1, x) + \mathrm{B}_1 \cos(N_1, y) + \mathrm{C}_1 \cos(N_1, z) \\
&\qquad - \mathrm{A}_2 \cos(N_2, x) - \mathrm{B}_2 \cos(N_2, y) - \mathrm{C}_2 \cos(N_2, z)] \\
&\qquad\qquad \times \left[\left(\frac{\partial V}{\partial x_1} - \frac{\partial V}{\partial x_2}\right)\delta x + \left(\frac{\partial V}{\partial y_1} - \frac{\partial V}{\partial y_2}\right)\delta y + \left(\frac{\partial V}{\partial z_1} - \frac{\partial V}{\partial z_2}\right)\delta z\right] dS_{12} \\
&+ \varepsilon \mathop{\mathrm{S}}_{S_{12}} [\mathrm{A}_1 \cos(N_1, x) + \mathrm{B}_1 \cos(N_1, y) + \mathrm{C}_1 \cos(N_1, z) \\
&\qquad + \mathrm{A}_2 \cos(N_2, x) + \mathrm{B}_2 \cos(N_2, y) + \mathrm{C}_2 \cos(N_2, z)]\, V \frac{dS_{12}' - dS_{12}}{dS_{12}} dS_{12}. \qquad (4)
\end{aligned}$$

Le symbole + etc. remplace neuf termes qui se déduisent, en permutant les indices 1 et 2, des neuf termes écrits avant ce symbole.

Nous allons transformer cette égalité.

Proposons nous d'abord d'évaluer la quantité $\delta \frac{\partial \mathrm{A}}{\partial x}$.

Prenons deux états voisins du système.

Soient $M(x, y, z)$ et $M_1(x_1, y, z)$ deux points infiniment voisins du système pris dans le premier état; la droite MM_1 est parallèle à l'axe des x. Aux deux points M, M_1 correspondent, dans le second état, deux points infiniment voisins $M'(x', y', z')$ et $M_1'(x_1', y_1', z_1')$; la droite $M'M_1'$ n'est plus, en général, parallèle à l'axe des x. Considérons les deux points $M_2'(x_2', y_2', z')$ et $M_3'(x_2', y', z')$. La droite $M'M_3'$ est parallèle à l'axe des x; la droite $M_3'M_2'$ est parallèle à l'axe des y; la droite $M_2'M_1'$ est parallèle à l'axe des z.

Soient A, A_1, A', A_1', A_2', A_3' les valeurs aux points M, M_1, M', M_1', M_2', M_3' de la composante parallèle à Ox de l'intensité de polarisation; les valeurs non accentuées se rapportent au premier état du système et les valeurs accentuées au second.

Formons la quantité

$$J = \frac{A'_3 - A'}{x'_1 - x'} - \frac{A_1 - A}{x_1 - x}.$$

Si, laissant invariables les deux états du système, nous faisons tendre le point M_1 vers le point M, le point M'_3 tendra en même temps vers le point M' et la quantité J tendra vers $\left(\frac{\partial A'}{\partial x'} - \frac{\partial A}{\partial x}\right)$. Si, ensuite, nous faisons tendre le second état du système vers le premier, cette différence deviendra une quantité infiniment petite dont le terme principal sera précisément $\delta \frac{\partial A}{\partial x}$.

Ce terme principal peut s'évaluer de la manière suivante :

On vérifie sans peine l'identité

$$\begin{aligned} \frac{A'_3 - A'}{x'_1 - x'} - \frac{A_1 - A}{x_1 - x} = {} & \frac{A'_3 - A'_2}{y'_3 - y'_2}\,\frac{y'_3 - y'_2}{x'_1 - x'} \\ & + \frac{A'_2 - A'_1}{z'_2 - z'_1}\,\frac{z'_2 - z'_1}{x'_1 - x'} \\ & - \frac{(A'_1 - A')[(x'_1 - x_1) - (x' - x)]}{(x_1 - x)(x'_1 - x')} \\ & + \frac{(A'_1 - A_1) - (A' - A)}{x_1 - x}. \end{aligned}$$

Les identités

$$y'_2 = y'_1, \quad y'_3 = y', \quad z'_2 = z',$$

permettent d'écrire le second membre de cette égalité sous la forme

$$\begin{aligned} & \frac{(A'_1 - A_1) - (A' - A)}{x_1 - x} \\ & - \frac{1}{1 + \frac{(x'_1 - x_1) - (x' - x)}{x_1 - x}} \left\{ \frac{A'_3 - A'_2}{y'_3 - y'_2}\,\frac{(y'_1 - y_1) - (y' - y)}{x_1 - x} \right. \\ & \qquad + \frac{A'_2 - A'_1}{z'_2 - z'_1}\,\frac{(z'_1 - z_1) - (z' - z)}{x_1 - x} \\ & \left. + \left[\frac{A_1 - A}{x_1 - x} - \frac{(A'_1 - A_1) - (A' - A)}{x_1 - x}\right] \frac{(x'_1 - x') - (x_1 - x)}{x_1 - x} \right\}. \end{aligned}$$

Laissons invariables les deux états du système et faisons tendre de point M_1 vers le point M. On voit sans peine que la quantité précédente tendra vers la limite

$$\frac{\partial (A'-A)}{\partial x} - \frac{1}{1+\frac{\partial (x'-x)}{\partial x}}\left\{\frac{\partial A'}{\partial y'}\frac{\partial (y'-y)}{\partial x} + \frac{\partial A'}{\partial z'}\frac{\partial (z'-z)}{\partial x} + \left[\frac{\partial A}{\partial x} - \frac{\partial (A'-A)}{\partial x}\right]\frac{\partial (x'-x)}{\partial x}\right\}.$$

Faisons maintenant tendre le second état du système vers le premier et cette quantité deviendra un infiniment petit dont le terme principal sera

$$\frac{\partial \delta A}{\partial x} - \frac{\partial A}{\partial y}\frac{\partial \delta y}{\partial x} - \frac{\partial A}{\partial z}\frac{\partial \delta z}{\partial x} - \frac{\partial A}{\partial x}\frac{\partial \delta x}{\partial x}.$$

On trouve ainsi la première des égalités

$$\left.\begin{aligned}
\delta\frac{\partial A}{\partial x} &= \frac{\partial \delta A}{\partial x} - \left(\frac{\partial A}{\partial x}\frac{\partial \delta x}{\partial x} + \frac{\partial A}{\partial y}\frac{\partial \delta y}{\partial x} + \frac{\partial A}{\partial z}\frac{\partial \delta z}{\partial x}\right),\\
\delta\frac{\partial B}{\partial y} &= \frac{\partial \delta B}{\partial y} - \left(\frac{\partial B}{\partial x}\frac{\partial \delta x}{\partial y} + \frac{\partial B}{\partial y}\frac{\partial \delta y}{\partial y} + \frac{\partial B}{\partial z}\frac{\partial \delta z}{\partial y}\right),\\
\delta\frac{\partial C}{\partial z} &= \frac{\partial \delta C}{\partial z} - \left(\frac{\partial C}{\partial x}\frac{\partial \delta x}{\partial z} + \frac{\partial C}{\partial y}\frac{\partial \delta y}{\partial z} + \frac{\partial C}{\partial z}\frac{\partial \delta z}{\partial z}\right).
\end{aligned}\right\} \qquad (5)$$

Les deux autres s'établissent d'une manière analogue.

Ces égalités (5) permettent d'écrire l'égalité

$$\begin{aligned}
&\int_1 V\delta\left(\frac{\partial A_1}{\partial x} + \frac{\partial B_1}{\partial y} + \frac{\partial C_1}{\partial z}\right)dv_1\\
&+\int_1 V\left(\frac{\partial A_1}{\partial x} + \frac{\partial B_1}{\partial y} + \frac{\partial C_1}{\partial z}\right)\left(\frac{\partial \delta x}{\partial x} + \frac{\partial \delta y}{\partial y} + \frac{\partial \delta z}{\partial z}\right)dv_1\\
&=\int_1 V\left(\frac{\partial \delta A_1}{\partial x} + \frac{\partial \delta B_1}{\partial y} + \frac{\partial \delta C_1}{\partial z}\right)dv_1\\
&+\int_1 V\left(\frac{\partial A_1}{\partial x}\frac{\partial \delta y}{\partial y} - \frac{\partial A_1}{\partial y}\frac{\partial \delta y}{\partial x} + \frac{\partial A_1}{\partial x}\frac{\partial \delta z}{\partial z} - \frac{\partial A_1}{\partial z}\frac{\partial \delta z}{\partial x}\right.\\
&\qquad + \frac{\partial B_1}{\partial y}\frac{\partial \delta z}{\partial z} - \frac{\partial B_1}{\partial z}\frac{\partial \delta z}{\partial y} + \frac{\partial B_1}{\partial y}\frac{\partial \delta x}{\partial x} - \frac{\partial B_1}{\partial x}\frac{\partial \delta x}{\partial y}\\
&\qquad \left. + \frac{\partial C_1}{\partial z}\frac{\partial \delta x}{\partial x} - \frac{\partial C_1}{\partial x}\frac{\partial \delta x}{\partial z} + \frac{\partial C_1}{\partial z}\frac{\partial \delta y}{\partial y} - \frac{\partial C_1}{\partial y}\frac{\partial \delta y}{\partial z}\right)dv_1.
\end{aligned}$$

Des intégrations par parties permettent de transformer cette égalité en la suivante :

$$
\begin{aligned}
&\int_1 V\delta\left(\frac{\partial \mathrm{A}_1}{\partial x}+\frac{\partial \mathrm{B}_1}{\partial y}+\frac{\partial \mathrm{C}_1}{\partial z}\right)dv_1\\
&+\int_1 V\left(\frac{\partial \mathrm{A}_1}{\partial x}+\frac{\partial \mathrm{B}_1}{\partial y}+\frac{\partial \mathrm{C}_1}{\partial z}\right)\left(\frac{\partial \delta x}{\partial x}+\frac{\partial \delta y}{\partial y}+\frac{\partial \delta z}{\partial z}\right)dv_1\\
&=-\int_1\left(\frac{\partial V}{\partial x}\delta\mathrm{A}_1+\frac{\partial V}{\partial y}\delta\mathrm{B}_1+\frac{\partial V}{\partial z}\delta\mathrm{C}_1\right)dv_1\\
&-\int_1\Big[\left(\frac{\partial \mathrm{B}_1}{\partial y}\frac{\partial V}{\partial x}-\frac{\partial \mathrm{B}_1}{\partial x}\frac{\partial V}{\partial y}+\frac{\partial \mathrm{C}_1}{\partial z}\frac{\partial V}{\partial x}-\frac{\partial \mathrm{C}_1}{\partial x}\frac{\partial V}{\partial z}\right)\delta x\\
&\qquad+\left(\frac{\partial \mathrm{C}_1}{\partial z}\frac{\partial V}{\partial y}-\frac{\partial \mathrm{C}_1}{\partial y}\frac{\partial V}{\partial z}+\frac{\partial \mathrm{A}_1}{\partial x}\frac{\partial V}{\partial y}-\frac{\partial \mathrm{A}_1}{\partial y}\frac{\partial V}{\partial x}\right)\delta y\\
&\qquad+\left(\frac{\partial \mathrm{A}_1}{\partial x}\frac{\partial V}{\partial z}-\frac{\partial \mathrm{A}_1}{\partial z}\frac{\partial V}{\partial x}+\frac{\partial \mathrm{B}_1}{\partial y}\frac{\partial V}{\partial z}-\frac{\partial \mathrm{B}_1}{\partial z}\frac{\partial V}{\partial y}\right)\delta z\Big]dv_1\\
&-\mathbf{S}_{S_1}V\left[\cos(N_i,x)\,\delta\mathrm{A}_1+\cos(N_i,y)\,\delta\mathrm{B}_1+\cos(N_i,z)\,\delta\mathrm{C}_1\right]dS_1\\
&-\mathbf{S}_{S_{12}}V\left[\cos(N_1,x)\,\delta\mathrm{A}_1+\cos(N_1,y)\,\delta\mathrm{B}_1+\cos(N_1,z)\,\delta\mathrm{C}_1\right]dS_{12}\\
&-\mathbf{S}_{S_1}V\left(\frac{\partial \mathrm{A}_1}{\partial x}+\frac{\partial \mathrm{B}_1}{\partial y}+\frac{\partial \mathrm{C}_1}{\partial z}\right)\left[\cos(N_i,x)\,\delta x+\cos(N_i,y)\,\delta y+\cos(N_i,z)\,\delta z\right]dS_1\\
&-\mathbf{S}_{S_{12}}V\left(\frac{\partial \mathrm{A}_1}{\partial x}+\frac{\partial \mathrm{B}_1}{\partial y}+\frac{\partial \mathrm{C}_1}{\partial z}\right)\left[\cos(N_1,x)\,\delta x+\cos(N_1,y)\,\delta y+\cos(N_1,z)\,\delta z\right]dS_{12}\\
&+\mathbf{S}_{S_1}V\Big[\left(\frac{\partial \mathrm{A}_1}{\partial x}\delta x+\frac{\partial \mathrm{A}_1}{\partial y}\delta y+\frac{\partial \mathrm{A}_1}{\partial z}\delta z\right)\cos(N_i,x)\\
&\qquad+\left(\frac{\partial \mathrm{B}_1}{\partial x}\delta x+\frac{\partial \mathrm{B}_1}{\partial y}\delta y+\frac{\partial \mathrm{B}_1}{\partial z}\delta z\right)\cos(N_i,y)\\
&\qquad+\left(\frac{\partial \mathrm{C}_1}{\partial x}\delta x+\frac{\partial \mathrm{C}_1}{\partial y}\delta y+\frac{\partial \mathrm{C}_1}{\partial z}\delta z\right)\cos(N_i,z)\Big]dS_1\\
&+\mathbf{S}_{S_{12}}V\Big[\left(\frac{\partial \mathrm{A}_1}{\partial x}\delta x+\frac{\partial \mathrm{A}_1}{\partial y}\delta y+\frac{\partial \mathrm{A}_1}{\partial z}\delta z\right)\cos(N_1,x)\\
&\qquad+\left(\frac{\partial \mathrm{B}_1}{\partial x}\delta x+\frac{\partial \mathrm{B}_1}{\partial y}\delta y+\frac{\partial \mathrm{B}_1}{\partial z}\delta z\right)\cos(N_1,y)\\
&\qquad+\left(\frac{\partial \mathrm{C}_1}{\partial x}\delta x+\frac{\partial \mathrm{C}_1}{\partial y}\delta y+\frac{\partial \mathrm{C}_1}{\partial z}\delta z\right)\cos(N_1,z)\Big]dS_{12}.
\end{aligned}
\tag{6}
$$

D'autre part, on a les identités suivantes :

$$\begin{aligned}
&\int_1 \left(\frac{\partial \mathrm{A}_1}{\partial x} + \frac{\partial \mathrm{B}_1}{\partial y} + \frac{\partial \mathrm{C}_1}{\partial z}\right)\left(\frac{\partial V}{\partial x}\,\delta x + \frac{\partial V}{\partial y}\,\delta y + \frac{\partial V}{\partial z}\,\delta z\right) dv_1 \\
&-\int_1 \Big[\left(\frac{\partial \mathrm{B}_1}{\partial y}\,\frac{\partial V}{\partial x} - \frac{\partial \mathrm{B}_1}{\partial x}\,\frac{\partial V}{\partial y} + \frac{\partial \mathrm{C}_1}{\partial z}\,\frac{\partial V}{\partial x} - \frac{\partial \mathrm{C}_1}{\partial x}\,\frac{\partial V}{\partial z}\right)\delta x \\
&\qquad + \left(\frac{\partial \mathrm{C}_1}{\partial z}\,\frac{\partial V}{\partial y} - \frac{\partial \mathrm{C}_1}{\partial y}\,\frac{\partial V}{\partial z} + \frac{\partial \mathrm{A}_1}{\partial x}\,\frac{\partial V}{\partial y} - \frac{\partial \mathrm{A}_1}{\partial y}\,\frac{\partial V}{\partial x}\right)\delta y \\
&\qquad + \left(\frac{\partial \mathrm{A}_1}{\partial x}\,\frac{\partial V}{\partial z} - \frac{\partial \mathrm{A}_1}{\partial z}\,\frac{\partial V}{\partial x} + \frac{\partial \mathrm{B}_1}{\partial y}\,\frac{\partial V}{\partial z} - \frac{\partial \mathrm{B}_1}{\partial z}\,\frac{\partial V}{\partial y}\right)\delta z\Big] dv_1 \\
&= \int_1 \Big[\left(\frac{\partial \mathrm{A}_1}{\partial x}\,\delta x + \frac{\partial \mathrm{A}_1}{\partial y}\,\delta y + \frac{\partial \mathrm{A}_1}{\partial z}\,\delta z\right)\frac{\partial V}{\partial x} \\
&\qquad + \left(\frac{\partial \mathrm{B}_1}{\partial x}\,\delta x + \frac{\partial \mathrm{B}_1}{\partial y}\,\delta y + \frac{\partial \mathrm{B}_1}{\partial z}\,\delta z\right)\frac{\partial V}{\partial y} \\
&\qquad + \left(\frac{\partial \mathrm{C}_1}{\partial x}\,\delta x + \frac{\partial \mathrm{C}_1}{\partial y}\,\delta y + \frac{\partial \mathrm{C}_1}{\partial z}\,\delta z\right)\frac{\partial V}{\partial z}\Big] dv_1
\end{aligned} \tag{7}$$

et

$$\begin{aligned}
&\mathrm{S}_{s_1} V\Big[\left(\frac{\partial \mathrm{A}_1}{\partial x}\,\delta x + \frac{\partial \mathrm{A}_1}{\partial y}\,\delta y + \frac{\partial \mathrm{A}_1}{\partial z}\,\delta z\right)\cos(N_i,\, x) \\
&\qquad + \left(\frac{\partial \mathrm{B}_1}{\partial z}\,\delta x + \frac{\partial \mathrm{B}_1}{\partial y}\,\delta y + \frac{\partial \mathrm{B}_1}{\partial z}\,\delta z\right)\cos(N_i,\, y) \\
&\qquad + \left(\frac{\partial \mathrm{C}_1}{\partial x}\,\delta x + \frac{\partial \mathrm{C}_1}{\partial y}\,\delta y + \frac{\partial \mathrm{C}_1}{\partial z}\,\delta z\right)\cos(N_i,\, z)\Big] dS_1 \\
&+ \mathrm{S}_{s_1} [\mathrm{A}_1 \cos(N_i,\, x) + \mathrm{B}_1 \cos(N_i,\, y) + \mathrm{C}_1 \cos(N_i,\, z)] \\
&\qquad \times \left(\frac{\partial V}{\partial x_i}\,\delta x + \frac{\partial V}{\partial y_i}\,\delta y + \frac{\partial V}{\partial z_i}\,\delta z\right) dS_1 \\
&= \mathrm{S}_{s_1} \Big\{\left[\frac{\partial}{\partial x_i}(\mathrm{A}_1 V)\,\delta x + \frac{\partial}{\partial y_i}(\mathrm{A}_1 V)\,\delta y + \frac{\partial}{\partial z_i}(\mathrm{A}_1 V)\,\delta z\right]\cos(N_i,\, x) \\
&\qquad + \left[\frac{\partial}{\partial x_i}(\mathrm{B}_1 V)\,\delta x + \frac{\partial}{\partial y_i}(\mathrm{B}_1 V)\,\delta y + \frac{\partial}{\partial z_i}(\mathrm{C}_1 V)\,\delta z\right]\cos(N_i,\, y) \\
&\qquad + \left[\frac{\partial}{\partial x_i}(\mathrm{C}_1 V)\,\delta x + \frac{\partial}{\partial y_i}(\mathrm{C}_1 V)\,\delta y + \frac{\partial}{\partial z_i}(\mathrm{C}_1 V)\,\delta z\right]\cos(N_i,\, z)\Big\} dS_1 .
\end{aligned} \tag{8}$$

On peut, en outre, écrire, en tout point de la surface S_{12}, une égalité analogue à la précédente ; nous la désignerons par (8 bis).

Enfin les égalités (32) du Chapître I, jointes à l'égalité (3) du présent Chapître donnent l'égalité :

$$\begin{aligned}\frac{\partial V}{\partial x_1}-\frac{\partial V}{\partial x_2} = {} & 4\pi\left[\mathrm{A}_1\cos(N_1, x)+\mathrm{B}_1\cos(N_1, y)+\mathrm{C}_1\cos(N_1, z)\right]\cos(N_1, x) \\ & +4\pi\left[\mathrm{A}_2\cos(N_2, x)+\mathrm{B}_2\cos(N_2, y)+\mathrm{C}_2\cos(N_2, z)\right]\cos(N_2, x), \end{aligned} \tag{9}$$

et deux égalités analogues pour $\left(\frac{\partial V}{\partial y_1}-\frac{\partial V}{\partial y_2}\right)$, $\left(\frac{\partial V}{\partial z_1}-\frac{\partial V}{\partial z_2}\right)$.

Moyennant les égalités (6), (7), (8), (8 bis) et (9), l'égalité (4) devient :

$$\begin{aligned}
-\delta Y = {} & -\varepsilon\int_1\left(\frac{\partial V}{\partial x}\delta\mathrm{A}_1+\frac{\partial V}{\partial y}\delta\mathrm{B}_1+\frac{\partial V}{\partial z}\delta\mathrm{C}_1\right)dv_1 \\
& +\varepsilon\int_1\Big[\frac{\partial V}{\partial x}\left(\frac{\partial\mathrm{A}_1}{\partial x}\delta x+\frac{\partial\mathrm{A}_1}{\partial y}\delta y+\frac{\partial\mathrm{A}_1}{\partial z}\delta z\right) \\
& \qquad +\frac{\partial V}{\partial y}\left(\frac{\partial\mathrm{B}_1}{\partial x}\delta x+\frac{\partial\mathrm{B}_1}{\partial y}\delta y+\frac{\partial\mathrm{B}_1}{\partial z}\delta z\right) \\
& \qquad +\frac{\partial V}{\partial z}\left(\frac{\partial\mathrm{C}_1}{\partial x}\delta x+\frac{\partial\mathrm{C}_1}{\partial y}\delta y+\frac{\partial\mathrm{C}_1}{\partial z}\delta z\right)\Big]dv_1 \\
& -\varepsilon\,\mathrm{S}_{S_1}V\left(\frac{\partial\mathrm{A}_1}{\partial x_i}+\frac{\partial\mathrm{B}_1}{\partial y_i}+\frac{\partial\mathrm{C}_1}{\partial z_i}\right) \\
& \qquad \times\left[\cos(N_i, x)\,\delta x+\cos(N_i, y)\,\delta y+\cos(N_i, z)\,\delta z\right]dS_1 \\
& +\varepsilon\,\mathrm{S}_{S_1}\Big\{\left[\frac{\partial(\mathrm{A}_1 V)}{\partial x_i}\delta x+\frac{\partial(\mathrm{A}_1 V)}{\partial y_i}\delta y+\frac{\partial(\mathrm{A}_1 V)}{\partial z_i}\delta z\right]\cos(N_i, x) \\
& \qquad +\left[\frac{\partial(\mathrm{B}_1 V)}{\partial x_i}\delta x+\frac{\partial(\mathrm{B}_1 V)}{\partial y_i}\delta y+\frac{\partial(\mathrm{B}_1 V)}{\partial z_i}\delta z\right]\cos(N_i, y) \\
& \qquad +\left[\frac{\partial(\mathrm{C}_1 V)}{\partial x_i}\delta x+\frac{\partial(\mathrm{C}_1 V)}{\partial y_i}\delta y+\frac{\partial(\mathrm{C}_1 V)}{\partial z_i}\delta z\right]\cos(N_i, z)\Big\}dS_1 \\
& +\varepsilon\,\mathrm{S}_{S_1}V\left[\mathrm{A}_1\delta\cos(N_i, x)+\mathrm{B}_1\delta\cos(N_i, y)+\mathrm{C}_1\delta\cos(N_i, z)\right]dS_1 \\
& +\varepsilon\,\mathrm{S}_{S_1}V\left[\mathrm{A}_1\cos(N_i, x)+\mathrm{B}_1\cos(N_i, y)+\mathrm{C}_1\cos(N_i, z)\right]\frac{dS_1'-dS_1}{dS_1}dS_1 \\
& -2\pi\varepsilon\,\mathrm{S}_{S_1}\left[\mathrm{A}_1\cos(N_i, x)+\mathrm{B}_1\cos(N_i, y)+\mathrm{C}_1\cos(N_i, z)\right]^2 \\
& \qquad \times\left[\cos(N_i, x)\,\delta x+\cos(N_i, y)\,\delta y+\cos(N_i, z)\,\delta z\right]dS_1 \\
& +\text{etc.}, \\
& -\varepsilon\,\mathrm{S}_{S_{12}}V\left(\frac{\partial\mathrm{A}_1}{\partial x_1}+\frac{\partial\mathrm{B}_1}{\partial y_1}+\frac{\partial\mathrm{C}_1}{\partial z_1}-\frac{\partial\mathrm{A}_2}{\partial x_2}-\frac{\partial\mathrm{B}_2}{\partial y_2}-\frac{\partial\mathrm{C}_2}{\partial z_2}\right) \\
& \qquad \times\left[\cos(N_1, x)\,\delta x+\cos(N_1, y)\,\delta y+\cos(N_1, z)\,\delta z\right]dS_{12}
\end{aligned}$$

$$
\begin{aligned}
&+\varepsilon \mathop{\mathrm{S}}_{S_{12}} \Big[\Big\{ \Big[\frac{\partial (\mathrm{A}_1 V)}{\partial x_1} - \frac{\partial (\mathrm{A}_2 V)}{\partial x_2} \Big] \delta x \\
&\qquad + \Big[\frac{\partial (\mathrm{A}_1 V)}{\partial y_1} + \frac{\partial (\mathrm{A}_2 V)}{\partial y_2} \Big] \delta y \\
&\qquad + \Big[\frac{\partial (\mathrm{A}_1 V)}{\partial z_1} - \frac{\partial (\mathrm{A}_2 V)}{\partial z_2} \Big] \delta z \Big\} \cos (N_1, x) \\
&\qquad + \Big\{ \Big[\frac{\partial (\mathrm{B}_1 V)}{\partial x_1} - \frac{\partial (\mathrm{B}_2 V)}{\partial x_2} \Big] \delta x \\
&\qquad + \Big[\frac{\partial (\mathrm{B}_1 V)}{\partial y_1} - \frac{\partial (\mathrm{B}_2 V)}{\partial y_2} \Big] \delta y \\
&\qquad + \Big[\frac{\partial (\mathrm{B}_1 V)}{\partial z_1} - \frac{\partial (\mathrm{B}_2 V)}{\partial z_2} \Big] \delta z \Big\} \cos (N_1, y) \\
&\qquad + \Big\{ \Big[\frac{\partial (\mathrm{C}_1 V)}{\partial x_1} - \frac{\partial (\mathrm{C}_2 V)}{\partial x_2} \Big] \delta x \\
&\qquad + \Big[\frac{\partial (\mathrm{C}_1 V)}{\partial y_1} - \frac{\partial (\mathrm{C}_2 V)}{\partial y_2} \Big] \delta y \\
&\qquad + \Big[\frac{\partial (\mathrm{C}_1 V)}{\partial z_1} - \frac{\partial (\mathrm{C}_2 V)}{\partial z_2} \Big] \delta z \Big\} \cos (N_1, z) \Big] \, dS_{12} \\
&+\varepsilon \mathop{\mathrm{S}}_{S_{12}} V [(\mathrm{A}_1 - \mathrm{A}_2)\, \delta \cos (N_1, x) + (\mathrm{B}_1 - \mathrm{B}_2)\, \delta \cos (N_1, y) \\
&\qquad\qquad + (\mathrm{C}_1 - \mathrm{C}_2)\, \delta \cos (N_1, z)] \, dS_{12} \\
&+\varepsilon \mathop{\mathrm{S}}_{S_{12}} V [(\mathrm{A}_1 - \mathrm{A}_2) \cos (N_1, x) + (\mathrm{B}_1 - \mathrm{B}_2) \cos (N_1, y) \\
&\qquad\qquad + (\mathrm{C}_1 - \mathrm{C}_2) \cos (N_1, z)] \, \frac{dS'_{12} - dS_{12}}{dS_{12}} \, dS_{12} \\
&-2\pi\varepsilon \mathop{\mathrm{S}}_{S_{12}} \{ [\mathrm{A}_1 \cos (N_1, x) + \mathrm{B}_1 \cos (N_1, y) + \mathrm{C}_1 \cos (N_1, z)]^2 \\
&\qquad - [\mathrm{A}_2 \cos (N_2, x) + \mathrm{B}_2 \cos (N_2, y) + \mathrm{C}_2 \cos (N_2, z)]^2 \} \\
&\qquad \times [\cos (N_1, x)\, \delta x + \cos (N_1, y)\, \delta y + \cos (N_1, z)\, \delta z] \, dS_{12}. \qquad (10)
\end{aligned}
$$

Le symbole + etc. remplace sept termes qui se déduisent des sept premiers termes de l'expression de δY en remplaçant l'indice 1 par l'indice 2.

Faisons choix, sur la surface S_1, d'un système de coordonnées curvilignes rectangulaires:

$$(u) \quad v = \text{const.},$$
$$(v) \quad u = \text{const.}$$

Supposons que le carré de l'élément linéaire tracé sur la surface soit représenté par la formule

$$ds_1^2 = A_1(u, v)\, du^2 + B_1(u, v)\, dv^2.$$

L'élément superficiel aura pour valeur

$$dS_1 = \sqrt{A_1 B_1}\, du\, dv. \tag{11}$$

Prenons un point M, intérieur au corps 1, pris dans le premier état, et infiniment voisin de la surface S_1; ce point a pour correspondant, dans le second état, un point M'. Projetons MM' sur la tangente à la ligne (u), sur la tangente à la ligne (v), enfin sur la normale N_i. Désignons les trois projections obtenues par

$$\sqrt{A}\,\delta u, \quad \sqrt{B}\,\delta v, \quad \delta N_i.$$

Nous aurons évidemment, pour un point (x_i, y_i, z_i) infiniment voisin de la surface S_1

$$\begin{aligned}\frac{\partial(\mathrm{A}_1 V)}{\partial x_i}\delta x + \frac{\partial(\mathrm{A}_1 V)}{\partial y_i}\delta y + \frac{\partial(\mathrm{A}_1 V)}{\partial z_i}\delta z \\ = \frac{\partial(\mathrm{A}_1 V)}{\partial u}\delta u + \frac{\partial(\mathrm{A}_1 V)}{\partial v}\delta v + \frac{\partial(\mathrm{A}_1 V)}{\partial N_i}\delta N_i\end{aligned} \tag{12}$$

et aussi

$$\delta \cos(N_i, x) = \frac{\partial \cos(N_i, x)}{\partial u}\delta u + \frac{\partial \cos(N_i, x)}{\partial v}\delta v + D\cos(N_i, x). \tag{13}$$

Le symbole $D \cos(N_i, x)$ a le sens suivant:

Soit M un point de la surface S_1; par ce point, menons la normale à la surface S_1; prolongeons la jusqu'au point m où elle rencontre la surface S_1'; en m menons la demi-normale n_i à la surface S_1' vers l'intérieur du fluide 1; nous aurons

$$D\cos(N_i, x) = \cos(n_i, x) - \cos(N_i, x). \tag{14}$$

Une égalité connue* donne

$$\begin{aligned}\frac{dS_1' - dS_1}{dS_1} = \frac{1}{2AB}\left[\frac{\partial(AB)}{\partial u}\delta u + \frac{\partial(AB)}{\partial v}\delta v\right] \\ + \frac{\delta du}{du} + \frac{\delta dv}{dv} - \left(\frac{1}{R_i} + \frac{1}{R_i'}\right)\delta N_i;\end{aligned} \tag{15}$$

R_i et R_i' sont les deux rayons de courbure principaux de la surface S_1, en un point de l'élément dS_1; chacun de ces rayons est compté positivement lorsque, pour aller de la surface au centre de courbure correspondant, on marche dans le sens de la demi-normale N_i.

---

* P. Duhem, *Hydrodynamique, élasticité, acoustique.* Tome II, p. 88.

Les égalités (11), (12), (13) et (15) nous permettent d'écrire

$$
\begin{aligned}
&\mathop{\mathrm{S}}_{S_1}\left[\frac{\partial(\mathrm{A}_1 V)}{\partial x_i}\delta x + \frac{\partial(\mathrm{A}_1 V)}{\partial y_i}\delta y + \frac{\partial(A_1 V)}{\partial z_i}\delta z\right]\cos(N_i, x)\,dS_1 \\
&+\mathop{\mathrm{S}}_{S_1}\mathrm{A}_1 V\,\delta\cos(N_i, x)\,dS_1 \\
&+\mathop{\mathrm{S}}_{S_1}\mathrm{A}_1 V\cos(N_i, x)\,\frac{dS_1' - dS_1}{dS_1}\,dS_1 \\
&=\mathop{\mathrm{S}}_{S_1}\left[\frac{\partial(\mathrm{A}_1 V)}{\partial N_i}\cos(N_i, x) - \mathrm{A}_1 V\left(\frac{1}{R_i} + \frac{1}{R_i'}\right)\cos(N_i, x)\right]\delta N_i dS_1 \\
&+\mathop{\mathrm{S}}_{S_1}\mathrm{A}_1 VD\cos(N_i, x)\,dS_1 \\
&+\int\!\!\int\Bigg[\left\{\frac{\partial\,[\mathrm{A}_1 V\cos(N_i, x)]}{\partial u} + \frac{\mathrm{A}_1 V\cos(N_i, x)}{2AB}\,\frac{\partial(AB)}{\partial u}\right\}\delta u \\
&\qquad + \left\{\frac{\partial\,[\mathrm{A}_1 V\cos(N_i, x)]}{\partial v} + \frac{\mathrm{A}_1 V\cos(N_i, x)}{2AB}\,\frac{\partial(AB)}{\partial v}\right\}\delta v \\
&\qquad\qquad + \mathrm{A}_1 V\cos(N_i, x)\left(\frac{\delta du}{du} + \frac{\delta dv}{dv}\right)\Bigg]\sqrt{AB}\,du\,dv. \qquad (16)
\end{aligned}
$$

Soient:

L_1 le contour de la surface S_1,

n_i une demi-droite normale à ce contour, tangente à la surface S_1, et dirigée vers l'intérieur de l'aire S_1,

(n_i, u) l'angle que cette demi-droite fait avec la tangente menée par son pied à la ligne (u) $(v = \text{const.})$, cette tangente étant dirigée dans le sens où le paramètre u va en croissant,

(n_i, v) l'angle que cette demi-droite fait avec la tangente menée par son pied à la ligne (v) $(u = \text{const.})$, cette tangente étant dirigée dans le sens où le paramètre v va en croissant,

$F(u, v)$ une fonction régulière des variables u, v.

Si la surface S_1 ne présente aucune singularité, on a*

$$
\begin{aligned}
&\int\!\!\int\Bigg\{\left[\frac{\partial F}{\partial u} + \frac{F}{2AB}\,\frac{\partial(AB)}{\partial u}\right]\delta u \\
&\qquad + \left[\frac{\partial F}{\partial v} + \frac{F}{2AB}\,\frac{\partial(AB)}{\partial v}\right]\delta v + F\left(\frac{\delta du}{du} + \frac{\delta dv}{dv}\right)\Bigg\}\sqrt{AB}\,du\,dv. \\
&+\int F\left[\cos(n_i, u)\sqrt{A}\,\delta u + \cos(n_i, v)\sqrt{B}\,\delta v\right]dL_1 = 0. \qquad (17)
\end{aligned}
$$

---

* P. Duhem, *Hydrodynamique, élasticité, acoustique.* T. II, p. 85.

Appliquons cette égalité (17) à la fonction

$$F = \mathrm{A}_1 V \cos(N_i, x);$$

remarquons, en outre, que l'on a

$$\cos(n_i, u)\sqrt{A}\,\delta u + \cos(n_i, v)\sqrt{B}\,\delta v = \cos(n_i, x)\,\delta x + \cos(n_i, y)\,\delta y + \cos(n_i, z)\,\delta z$$

et l'égalité (16) se transformera en une autre égalité qui, jointe à deux égalités analogues, donnera

$$\begin{aligned}
&\mathrm{S}_{S_1}\Big\{\Big[\frac{\partial(\mathrm{A}_1V)}{\partial x_i}\delta x + \frac{\partial(\mathrm{A}_1V)}{\partial y_i}\delta y + \frac{\partial(\mathrm{A}_1V)}{\partial z_i}\delta z\Big]\cos(N_i, x)\\
&\quad + \Big[\frac{\partial(\mathrm{B}_1V)}{\partial x_i}\delta x + \frac{\partial(\mathrm{B}_1V)}{\partial y_i}\delta y + \frac{\partial(\mathrm{B}_1V)}{\partial z_i}\delta z\Big]\cos(N_i, y)\\
&\quad + \Big[\frac{\partial(\mathrm{C}_1V)}{\partial x_i}\delta x + \frac{\partial(\mathrm{C}_1V)}{\partial y_i}\delta y + \frac{\partial(C_1V)}{\partial z_i}\delta z\Big]\cos(N_i, z)\Big\}\,dS_1\\
&+ \mathrm{S}_{S_1} V[\mathrm{A}_1\delta\cos(N_i, x) + \mathrm{B}_1\delta\cos(N_i, y) + \mathrm{C}_1\delta\cos(N_i, z)]\,dS_1\\
&+ \mathrm{S}_{S_1} V[\mathrm{A}_1\cos(N_i, x) + \mathrm{B}_1\cos(N_i, y) + C_1\cos(N_i, z)]\,\frac{dS_1' - dS_1}{dS_1}\,dS_1\\
&= \mathrm{S}_{S_1} V[\mathrm{A}_1D\cos(N_i, x) + \mathrm{B}_1D\cos(N_i, y) + \mathrm{C}_1D\cos(N_i, z)]\,dS_1\\
&+ \mathrm{S}_{S_1}\Big\{\frac{\partial(\mathrm{A}_1V)}{\partial N_i}\cos(N_i, x) + \frac{\partial(\mathrm{B}_1V)}{\partial N_i}\cos(N_i, y) + \frac{\partial(\mathrm{C}_1V)}{\partial N_i}\cos(N_i, z)\\
&\qquad - V[\mathrm{A}_1\cos(N_i, x) + \mathrm{B}_1\cos(N_i, y) + \mathrm{C}_1\cos(N_i, z)]\Big(\frac{1}{R_i} + \frac{1}{R_i'}\Big)\Big\}\\
&\qquad \times [\cos(N_i, x)\,\delta x + \cos(N_i, y)\,\delta y + \cos(N_i, z)\,\delta z]\,dS_1\\
&- \int_{L_1} V[\mathrm{A}_1\cos(N_i, x) + \mathrm{B}_1\cos(N_i, y) + \mathrm{C}_1\cos(N_i, z)]\\
&\qquad \times [\cos(n_i, x)\,\delta x + \cos(n_i, y)\,\delta y + \cos(n_i, z)\,\delta z]\,dL_1. \qquad (18)
\end{aligned}$$

Proposons nous maintenant d'évaluer la quantité

$$\mathrm{S}_{S_1} V[\mathrm{A}_1D\cos(N_i, x) + \mathrm{B}_1D\cos(N_i, y) + \mathrm{C}_1D\cos(N_i, z)]\,dS_1,$$

et, pour cela, cherchons d'abord l'expression de $D\cos(N_i, x)$ en un point quelconque M_1 de la surface S_1.

Soient: Σ_1 une aire quelconque tracée sur la surface S_1, autour du point M_1,

λ_1 le contour de l'aire Σ_1,

ν_i une demi-droite normale à ce contour, tangente à la surface S_1 et dirigée vers l'intérieur de l'aire Σ_1.

Par chaque point M de l'aire Σ_1, élevons une normale à la surface S_1, et soit m le point où cette normale rencontre la surface S_1'; à chaque point M, faisons correspondre le point m ainsi défini; à l'aire Σ_1, tracée sur la surface S_1, correspondra une aire σ_1 tracée sur la surface S_1'; $d\Sigma_1$, $d\sigma_1$, seront deux éléments correspondants des aires Σ_1, σ_1.

Un lemme bien connu de Gauss nous apprend que, pour une surface fermée quelconque S, on a

$$\mathop{\mathrm{S}} \cos(N, x)\, dS = 0,$$

pourvu que les demi-normales N soient portées toutes vers l'intérieur, ou toutes vers l'extérieur de la surface S.

Appliquons ce lemme à la surface fermée que composent l'aire Σ_1, l'aire σ_1, et la surface réglée engendrée par les normales à la surface S_1 le long du contour λ_1; il est facile de voir que nous aurons, en tout état de cause:

$$\mathop{\mathrm{S}}_{\sigma_1} \cos(n_i, x)\, d\sigma_1 - \mathop{\mathrm{S}}_{\Sigma_1} \cos(N_i, x)\, d\Sigma_1 + \int \cos(\nu_i, x)\, \delta N_i\, d\lambda_1 = 0,$$

ou bien, en tenant compte de l'égalité (14) et en remarquant que

$$\frac{d\sigma_1 - d\Sigma_1}{d\Sigma_1} = -\left(\frac{1}{R_i} + \frac{1}{R_i'}\right) \delta N_i,$$

$$\mathop{\mathrm{S}}_{\Sigma_1} D \cos(N_i, x)\, d\Sigma_1 = \mathop{\mathrm{S}}_{\Sigma_1} \left(\frac{1}{R_i} + \frac{1}{R_i'}\right) \cos(N_i, x)\, \delta N_i\, d\Sigma_1 - \int \cos(\nu_i, x)\, \delta N_i\, d\lambda_1. \tag{19}$$

Les égalités

$$\left.\begin{aligned}
&\cos(\nu_i, x) \frac{\partial x}{\partial u} + \cos(\nu_i, y) \frac{\partial y}{\partial u} + \cos(\nu_i, z) \frac{\partial z}{\partial u} = \sqrt{A} \cos(\nu_i, u), \\
&\cos(\nu_i, x) \frac{\partial x}{\partial v} + \cos(\nu_i, y) \frac{\partial y}{\partial v} + \cos(\nu_i, z) \frac{\partial z}{\partial v} = \sqrt{B} \cos(\nu_i, v), \\
&\cos(\nu_i, x) \cos(N_i, x) + \cos(\nu_i, y) \cos(N_i, y) + \cos(\nu_i, z) \cos(N_i, z) = 0,
\end{aligned}\right\} \tag{20}$$

donnent:

$$\begin{aligned}
\cos(\nu_i, x) = \frac{1}{\mathfrak{D}} \Big\{ &\Big[\frac{\partial y}{\partial v} \cos(N_i, z) - \frac{\partial z}{\partial v} \cos(N_i, y)\Big] \sqrt{A} \cos(\nu_i, u) \\
&- \Big[\frac{\partial y}{\partial u} \cos(N_i, z) - \frac{\partial z}{\partial u} \cos(N_i, y)\Big] \sqrt{B} \cos(\nu_i, v) \Big\},
\end{aligned} \tag{21}$$

en posant :

$$\mathfrak{D} = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial y}{\partial u} & \dfrac{\partial z}{\partial u} \\ \dfrac{\partial x}{\partial v} & \dfrac{\partial y}{\partial v} & \dfrac{\partial z}{\partial v} \\ \cos(N_i, x) & \cos(N_i, y) & \cos(N_i, z) \end{vmatrix}. \tag{22}$$

Nous aurons alors

$$\begin{aligned} &\int \cos(\nu_i, x)\, \delta N_i\, d\lambda_1 \\ &\quad = \int \frac{\delta N_i}{\mathfrak{D}} \Big\{ \Big[\frac{\partial y}{\partial v} \cos(N_i, z) - \frac{\partial z}{\partial v} \cos(N_i, y)\Big] \sqrt{A} \cos(\nu_i, u) \\ &\qquad - \Big[\frac{\partial y}{\partial v} \cos(N_i, z) - \frac{\partial z}{\partial u} \cos(N_i, y)\Big] \sqrt{B} \cos(\nu_i, v) \Big\} d\lambda_1. \end{aligned} \tag{23}$$

Mais si $G(u, v)$ désigne une fonction de u, v, régulière dans l'aire Σ_1, on a

$$\left.\begin{aligned} \int G\sqrt{A} \cos(\nu_i, u)\, d\lambda_1 &= \int\!\!\int \frac{\partial G}{\partial u} \sqrt{AB}\, du\, dv = \mathrm{S}_{\Sigma_1} \frac{\partial G}{\partial u}\, d\Sigma_1, \\ \int G\sqrt{B} \cos(\nu_i, v)\, d\lambda_1 &= \int\!\!\int \frac{\partial G}{\partial v} \sqrt{AB}\, du\, dv = \mathrm{S}_{\Sigma_1} \frac{\partial G}{\partial v}\, d\Sigma_1. \end{aligned}\right\} \tag{24}$$

Moyennant ces lemmes, l'égalité (23) peut s'écrire

$$\begin{aligned} \int \cos(\nu_i, x)\, \delta N_i\, d\lambda_1 = \mathrm{S}_{\Sigma_1} \Big[&\frac{\partial}{\partial u} \Big\{ \frac{\delta N_i}{\mathfrak{D}} \Big[\frac{\partial y}{\partial v} \cos(N_i, z) - \frac{\partial z}{\partial v} \cos(N_i, y)\Big] \Big\} \\ &- \frac{\partial}{\partial v} \Big\{ \frac{\delta N_i}{\mathfrak{D}} \Big[\frac{\partial y}{\partial v} \cos(N_i, z) - \frac{\partial z}{\partial u} \cos(N_i, y)\Big] \Big\} \Big] d\Sigma_1. \end{aligned} \tag{25}$$

Moyennant cette égalité (25), l'égalité (29) devient

$$\begin{aligned} \mathrm{S}_{\Sigma_1} \Big[&D \cos(N_i, x) - \Big(\frac{1}{R_i} + \frac{1}{R_i'}\Big) \cos(N_i, x)\, \delta N_i \\ &+ \frac{\partial}{\partial u} \Big\{ \frac{\delta N_i}{\mathfrak{D}} \Big[\frac{\partial y}{\partial v} \cos(N_i, z) - \frac{\partial z}{\partial v} \cos(N_i, y)\Big] \Big\} \\ &- \frac{\partial}{\partial v} \Big\{ \frac{\delta N_i}{\mathfrak{D}} \Big[\frac{\partial y}{\partial u} \cos(N_i, z) - \frac{\partial z}{\partial u} \cos(N_i, y)\Big] \Big\} \Big] d\Sigma_1 = 0. \end{aligned}$$

Cette égalité doit avoir lieu quelle que soit l'aire Σ_1, tracée autour du point M_1 ; on en conclut sans peine que l'on doit avoir, en tout point M_1 de la surface S_1,

$$\begin{aligned} D \cos(N_i, x) = &\Big(\frac{1}{R_i} + \frac{1}{R_i'}\Big) \cos(N_i, x)\, \delta N_i \\ &- \frac{\partial}{\partial u} \Big\{ \frac{\delta N_i}{\mathfrak{D}} \Big[\frac{\partial y}{\partial v} \cos(N_i, z) - \frac{\partial z}{\partial v} \cos(N_i, y)\Big] \Big\} \\ &+ \frac{\partial}{\partial v} \Big\{ \frac{\delta N_i}{\mathfrak{D}} \Big[\frac{\partial y}{\partial u} \cos(N_i, z) - \frac{\partial z}{\partial u} \cos(N_i, y)\Big] \Big\}. \end{aligned} \tag{26}$$

Cette égalité (26) nous permet d'écrire:

$$\begin{aligned}\mathop{\mathrm{S}}\nolimits_{S_1} V\mathrm{A}_1 D\cos(N_i,\,x)\,dS_1 &= \mathop{\mathrm{S}}\nolimits_{S_1} V\mathrm{A}_1\cos(N_i,\,x)\Big(\frac{1}{R_i}+\frac{1}{R_i'}\Big)\delta N_i\,dS_1\\ &-\mathop{\mathrm{S}}\nolimits_{S_1}\Big[V\mathrm{A}_1\frac{\partial}{\partial u}\Big\{\frac{\delta N_i}{\mathfrak{D}}\Big[\frac{\partial y}{\partial v}\cos(N_i,\,z)-\frac{\partial z}{\partial v}\cos(N_i,\,y)\Big]\Big\}\\ &\qquad -V\mathrm{A}_1\frac{\partial}{\partial v}\Big\{\frac{\delta N_i}{\mathfrak{D}}\Big[\frac{\partial y}{\partial u}\cos(N_i,\,z)-\frac{\partial z}{\partial u}\cos(N_i,\,y)\Big]\Big\}\Big]\,dS_1.\end{aligned}\tag{27}$$

Mais on a

$$\begin{aligned}&\mathop{\mathrm{S}}\nolimits_{S_1}\Big[V\mathrm{A}_1\frac{\partial}{\partial u}\Big\{\frac{\delta N_i}{\mathfrak{D}}\Big[\frac{\partial y}{\partial v}\cos(N_i,\,z)-\frac{\partial z}{\partial v}\cos(N_i,\,y)\Big]\Big\}\\ &\qquad -V\mathrm{A}_1\frac{\partial}{\partial v}\Big\{\frac{\delta N_i}{\mathfrak{D}}\Big[\frac{\partial y}{\partial u}\cos(N_i,\,z)-\frac{\partial z}{\partial u}\cos(N_i,\,y)\Big]\Big\}\Big]\,dS_1\\ &=\mathop{\mathrm{S}}\nolimits_{S_1}\Big[\frac{\partial}{\partial u}\Big\{\frac{V\mathrm{A}_1\delta N_i}{\mathfrak{D}}\Big[\frac{\partial y}{\partial v}\cos(N_i,\,z)-\frac{\partial z}{\partial v}\cos(N_i,\,y)\Big]\Big\}\\ &\qquad -\frac{\partial}{\partial v}\Big\{\frac{V\mathrm{A}_1\delta N_i}{\mathfrak{D}}\Big[\frac{\partial y}{\partial u}\cos(N_i,\,z)-\frac{\partial z}{\partial u}\cos(N_i,\,y)\Big]\Big\}\Big]\,dS_1\\ &-\mathop{\mathrm{S}}\nolimits_{S_1}\Big\{\frac{1}{\mathfrak{D}}\Big[\frac{\partial y}{\partial v}\cos(N_i,\,z)-\frac{\partial z}{\partial v}\cos(N_i,\,y)\Big]\frac{\partial(\mathrm{A}_1V)}{\partial u}\\ &\qquad -\frac{1}{\mathfrak{D}}\Big[\frac{\partial y}{\partial u}\cos(N_i,\,z)-\frac{\partial z}{\partial u}\cos(N_i,\,y)\Big]\frac{\partial(\mathrm{A}_1V)}{\partial v}\Big\}\,\delta N_i\,dS_1.\end{aligned}\tag{28}$$

D'ailleurs les égalités

$$\cos(u,\,x)\frac{\partial x}{\partial u}+\cos(u,\,y)\frac{\partial y}{\partial u}+\cos(u,\,z)\frac{\partial z}{\partial u}=\sqrt{\mathrm{A}},$$

$$\cos(u,\,x)\frac{\partial x}{\partial v}+\cos(u,\,y)\frac{\partial y}{\partial v}+\cos(u,\,z)\frac{\partial z}{\partial v}=0,$$

$$\cos(u,\,x)\cos(N_i,\,x)+\cos(u,\,y)\cos(N_i,\,y)+\cos(u,\,z)\cos(N_i,\,z)=0,$$

qui résultent immédiatement des égalités:

$$\frac{\partial x}{\partial u}=\sqrt{A}\cos(u,\,x),\quad \frac{\partial y}{\partial u}=\sqrt{A}\cos(u,\,y),\quad \frac{\partial z}{\partial u}=\sqrt{A}\cos(u,\,z),$$

$$\frac{\partial x}{\partial v}=\sqrt{B}\cos(v,\,x),\quad \frac{\partial y}{\partial v}=\sqrt{B}\cos(v,\,y),\quad \frac{\partial z}{\partial v}=\sqrt{B}\cos(v,\,z),$$

donnent:

$$\frac{\cos(u,\,x)}{\sqrt{A}}=\frac{1}{\mathfrak{D}}\Big[\frac{\partial y}{\partial v}\cos(N_i,\,z)-\frac{\partial z}{\partial v}\cos(N_i,\,y)\Big].$$

On trouve de même

$$\frac{\cos(v,x)}{\sqrt{B}} = -\frac{1}{\mathfrak{D}}\left[\frac{\partial y}{\partial u}\cos(N_i,z) - \frac{\partial z}{\partial u}\cos(N_i,y)\right].$$

Moyennant ces égalités, on a

$$\begin{aligned}\mathrm{S}_{S_1}\Big\{&\frac{1}{\mathfrak{D}}\left[\frac{\partial y}{\partial v}\cos(N_i,z) - \frac{\partial z}{\partial v}\cos(N_i,y)\right]\frac{\partial(\mathrm{A}_1V)}{\partial u}\\ &-\frac{1}{\mathfrak{D}}\left[\frac{\partial y}{\partial u}\cos(N_i,z) - \frac{\partial z}{\partial u}\cos(N_i,y)\right]\frac{\partial(\mathrm{A}_1V)}{\partial v}\Big\}\,\delta N_i\,dS_1\\ = \mathrm{S}_{S_1}&\left[\frac{1}{\sqrt{A}}\frac{\partial(\mathrm{A}_1V)}{\partial u}\cos(u,x) + \frac{1}{\sqrt{B}}\frac{\partial(\mathrm{A}_1V)}{\partial v}\cos(v,x)\right]\delta N_i\,dS_1.\end{aligned}$$

Mais, d'autre part, on a

$$\frac{1}{\sqrt{A}}\frac{\partial(\mathrm{A}_1V)}{\partial u}\cos(u,x) + \frac{1}{\sqrt{B}}\frac{\partial(\mathrm{A}_1V)}{\partial v}\cos(v,x) + \frac{\partial(\mathrm{A}_1V)}{\partial N_i}\cos(N_i,x) = \frac{\partial(\mathrm{A}_1V)}{\partial x_i},$$

en sorte que l'égalité précédente devient

$$\begin{aligned}\mathrm{S}_{S_1}\Big\{&\frac{1}{\mathfrak{D}}\left[\frac{\partial y}{\partial v}\cos(N_i,z) - \frac{\partial z}{\partial v}\cos(N_i,y)\right]\frac{\partial(\mathrm{A}_1V)}{\partial u}\\ &-\frac{1}{\mathfrak{D}}\left[\frac{\partial y}{\partial u}\cos(N_i,z) - \frac{\partial z}{\partial u}\cos(N_i,y)\right]\frac{\partial(\mathrm{A}_1V)}{\partial v}\Big\}\,\delta N_i\,dS_1\\ = \mathrm{S}_{S_1}&\left[\frac{\partial(\mathrm{A}_1V)}{\partial x_i} - \frac{\partial(\mathrm{A}_1V)}{\partial N_i}\cos(N_i,x)\right]\delta N_i\,dS_1.\end{aligned} \tag{29}$$

Appliquons maintenant à la surface S_1 tout entière le lemme représenté par les identités (21); nous aurons

$$\begin{aligned}\mathrm{S}_{S_1}\Big[&\frac{\partial}{\partial u}\left\{\frac{\mathrm{A}_1V\delta N_i}{\mathfrak{D}}\left[\frac{\partial y}{\partial v}\cos(N_i,z) - \frac{\partial z}{\partial v}\cos(N_i,y)\right]\right\}\\ &-\frac{\partial}{\partial v}\left\{\frac{\mathrm{A}_1V\delta N_i}{\mathfrak{D}}\left[\frac{\partial y}{\partial u}\cos(N_i,z) - \frac{\partial z}{\partial u}\cos(N_i,y)\right]\right\}\Big]dS_1\\ = \int \mathrm{A}_1V\Big\{&\frac{1}{\mathfrak{D}}\left[\frac{\partial y}{\partial v}\cos(N_i,z) - \frac{\partial z}{\partial v}\cos(N_i,y)\right]\sqrt{A}\cos(n_i,u)\\ &-\frac{1}{\mathfrak{D}}\left[\frac{\partial y}{\partial u}\cos(N_i,z) - \frac{\partial z}{\partial v}\cos(N_i,y)\right]\sqrt{B}\cos(n_i,v)\Big\}\,\delta N_i\,dL_1.\end{aligned}$$

Les égalités

$$\cos(u, x) = \frac{\sqrt{A}}{\mathfrak{D}}\left[\frac{\partial y}{\partial v}\cos(N_i, z) - \frac{\partial z}{\partial v}\cos(N_i, y)\right]$$

$$\cos(v, x) = -\frac{\sqrt{B}}{\mathfrak{D}}\left[\frac{\partial y}{\partial u}\cos(N_i, z) - \frac{\partial z}{\partial v}\cos(N_i, y)\right]$$

transforment cette égalité en

$$\begin{aligned}
&\mathop{\mathrm{S}}\nolimits_{s_1}\Bigg[\frac{\partial}{\partial u}\left\{\frac{\mathrm{A}_1 V\delta N_i}{\mathfrak{D}}\left[\frac{\partial y}{\partial v}\cos(N_i, z) - \frac{\partial z}{\partial v}\cos(N_i, y)\right]\right\} \\
&\qquad - \frac{\partial}{\partial v}\left\{\frac{\mathrm{A}_1 V\delta N_i}{\mathfrak{D}}\left[\frac{\partial y}{\partial u}\cos(N_i, z) - \frac{\partial z}{\partial u}\cos(N_i, y)\right]\right\}\Bigg] dS_1 \\
&= \int \mathrm{A}_1 V\left[\cos(n_i, u)\cos(u, x) + \cos(n_i, v)\cos(v, x)\right] bN_i\, dL_1 \\
&= \int \mathrm{A}_1 \dot{V}\cos(n_i, x)\,\delta N_i\, dL_1. \qquad (30)
\end{aligned}$$

Les égalités (27), (28), (29), (30) donnent

$$\begin{aligned}
\mathop{\mathrm{S}}\nolimits_{s_1}\mathrm{A}_1 VD\cos(N_i, x)\, dS_1 = {} & \mathop{\mathrm{S}}\nolimits_{s_1}\mathrm{A}_1 V\cos(N_i, x)\left(\frac{1}{R_i} + \frac{1}{R_i'}\right)\delta N_i\, dS_1 \\
& - \mathop{\mathrm{S}}\nolimits_{s_1}\left[\frac{\partial(\mathrm{A}_1 V)}{\partial N_i}\cos(N_i, x) - \frac{\partial(\mathrm{A}_1 V)}{\partial x_i}\right]\delta N_i\, dS_1 \\
& + \int \mathrm{A}_1 V\cos(n_i, x)\,\delta N_i\, dL_1.
\end{aligned}$$

Cette égalité, jointe à deux autres égalités semblables que l'on obtient en permutant la lettre x avec les lettres y et z, donne

$$\begin{aligned}
&\mathop{\mathrm{S}}\nolimits_{s_1} V\left[\mathrm{A}_1 D\cos(N_i, x) + \mathrm{B}_1 D\cos(N_i, y) + \mathrm{C}_1 D\cos(N_i, z)\right] dS_1 \\
&= \mathop{\mathrm{S}}\nolimits_{s_1} V\left[\mathrm{A}_1\cos(N_i, x) + \mathrm{B}_1\cos(N_i, y) + \mathrm{C}_1\cos(N_i, z)\right]\left(\frac{1}{R_i} + \frac{1}{R_i'}\right)\delta N_i\, dS_1 \\
&- \mathop{\mathrm{S}}\nolimits_{s_1}\left[\frac{\partial(\mathrm{A}_1 V)}{\partial N_i}\cos(N_i, x) + \frac{\partial(\mathrm{B}_1 V)}{\partial N_i}\cos(N_i, y) + \frac{\partial(\mathrm{C}_1 V)}{\partial N_i}\cos(N_i, z)\right]\delta N_i\, dS_1 \\
&+ \mathop{\mathrm{S}}\nolimits_{s_1}\left[\frac{\partial(\mathrm{A}_1 V)}{\partial x_i} + \frac{\partial(\mathrm{B}_1 V)}{\partial y_i} + \frac{\partial(\mathrm{C}_1 V)}{\partial z_i}\right]\delta N_i\, dS_1 \\
&+ \int V\left[\mathrm{A}_1\cos(n_i, x) + \mathrm{B}_1\cos(n_i, y) + \mathrm{C}_1\cos(n_i, z)\right]\delta N_i\, dL_1. \qquad (31)
\end{aligned}$$

Les égalités (18) et (31) donnent:

$$
\begin{aligned}
&\mathrm{S}_{S_1}\Big\{\Big[\frac{\partial(\mathrm{A}_1 V)}{\partial x_i}\delta x+\frac{\partial(\mathrm{A}_1 V)}{\partial y_i}\delta y+\frac{\partial(\mathrm{A}_1 V)}{\partial z_i}\delta z\Big]\cos(N_i,x)\\
&\quad+\Big[\frac{\partial(\mathrm{B}_1 V)}{\partial x_i}\delta x+\frac{\partial(\mathrm{B}_1 V)}{\partial y_i}\delta y+\frac{\partial(\mathrm{B}_1 V)}{\partial z_i}\delta z\Big]\cos(N_i,y)\\
&\quad+\Big[\frac{\partial(\mathrm{C}_1 V)}{\partial x_i}\delta x+\frac{\partial(\mathrm{C}_1 V)}{\partial y_i}\delta y+\frac{\partial(\mathrm{C}_1 V)}{\partial z_i}\delta z\Big]\cos(N_i,z)\Big\}dS_1\\
&+\mathrm{S}_{S_1}V[\mathrm{A}_1\delta\cos(N_i,x)+\mathrm{B}_1\delta\cos(N_i,y)+\mathrm{C}_1\delta\cos(N_i,z)]\,dS_1\\
&+\mathrm{S}_{S_1}V[\mathrm{A}_1\cos(N_i,x)+\mathrm{B}_1\cos(N_i,y)+\mathrm{C}_1\cos(N_i,z)]\,\frac{dS_1'-dS_1}{dS_1}\,dS_1\\
&=\mathrm{S}_{S_1}\Big[\frac{\partial(\mathrm{A}_1 V)}{\partial x_i}+\frac{\partial(\mathrm{B}_1 V)}{\partial y_i}+\frac{\partial(\mathrm{C}_1 V)}{\partial z_i}\Big]\delta N_i\,dS_1\\
&+\int V\{[\mathrm{A}_1\cos(n_i,x)+\mathrm{B}_1\cos(n_i,y)+\mathrm{C}_1\cos(n_i,z)]\,dN_i\\
&\quad-[\mathrm{A}_1\cos(N_i,x)+\mathrm{B}_1\cos(N_i,y)+\mathrm{C}_1\cos(N_i,z)]\,\delta n_i\}\,dL_1.
\end{aligned}
\tag{32}
$$

Cette égalité est générale.

Mais l'expression de δY obtenue au Chapître I cesserait d'être valable si, dans la déformation du système, les lignes le long desquelles la densité σ est discontinue se déplaçaient; *on ne doit donc appliquer les formules établies qu'aux déplacements pour lesquels*

$$\delta x=0,\quad \delta y=0,\quad \delta z=0,$$

en tout point des lignes L_1, L_2, L_{12}, *qui limitent les surfaces* S_1, S_2, S_{12}.

Dans ces conditions, l'égalité (32) peut s'écrire simplement

$$
\begin{aligned}
&\mathrm{S}_{S_1}\Big\{\Big[\frac{\partial(\mathrm{A}_1 V)}{\partial x_i}\delta x+\frac{\partial(\mathrm{A}_1 V)}{\partial y_i}\delta y+\frac{\partial(\mathrm{A}_1 V)}{\partial z_i}\delta z\Big]\cos(N_i,x)\\
&\quad+\Big[\frac{\partial(\mathrm{B}_1 V)}{\partial x_i}\delta x+\frac{\partial(\mathrm{B}_1 V)}{\partial y_i}\delta y+\frac{\partial(\mathrm{B}_1 V)}{\partial z_i}\delta z\Big]\cos(N_i,y)\\
&\quad+\Big[\frac{\partial(\mathrm{C}_1 V)}{\partial x_i}\delta x+\frac{\partial(\mathrm{C}_1 V)}{\partial y_i}\delta y+\frac{\partial(\mathrm{C}_1 V)}{\partial z_i}\delta z\Big]\cos(N_i,z)\Big\}dS_1\\
&+\mathrm{S}_{S_1}V[\mathrm{A}_1\delta\cos(N_i,x)+\mathrm{B}_1\delta\cos(N_i,y)+\mathrm{C}_1\delta\cos(N_i,z)]\,dS_1\\
&+\mathrm{S}_{S_1}V[\mathrm{A}_1\cos(N_i,x)+\mathrm{B}_1\cos(N_i,y)+\mathrm{C}_1\cos(N_i,z)]\,\frac{dS_1'-dS_1}{dS_1}\,dS_1\\
&=\mathrm{S}_{S_1}\Big[\frac{\partial(\mathrm{A}_1 V)}{\partial x_i}+\frac{\partial(\mathrm{B}_1 V)}{\partial y_i}+\frac{\partial(\mathrm{C}_1 V)}{\partial z_i}\Big]\delta N_i\,dS_1.
\end{aligned}
\tag{33}
$$

Nous pouvons écrire, pour la surface S_2, une égalité analogue ; nous la désignerons par (33 bis) ; enfin une démonstration semblable nous permettra d'écrire

$$\begin{aligned}
&\mathop{\mathrm{S}}_{S_{12}}\Big[\Big\{\Big[\frac{\partial(\mathrm{A}_1 V)}{\partial x_1}-\frac{\partial(\mathrm{A}_2 V)}{\partial x_2}\Big]\delta x+\Big[\frac{\partial(\mathrm{A}_1 V)}{\partial y_1}-\frac{\partial(\mathrm{A}_2 V)}{\partial y_2}\Big]\delta y \\
&\qquad\qquad+\Big[\frac{\partial(\mathrm{A}_1 V)}{\partial z_1}-\frac{\partial(\mathrm{A}_2 V)}{\partial z_2}\Big]\delta z\Big\}\cos(N_i, x) \\
&\quad+\Big\{\Big[\frac{\partial(\mathrm{B}_1 V)}{\partial x_1}-\frac{\partial(\mathrm{B}_2 V)}{\partial x_2}\Big]\delta x+\Big[\frac{\partial(\mathrm{B}_1 V)}{\partial y_1}-\frac{\partial(\mathrm{B}_2 V)}{\partial y_2}\Big]\delta y \\
&\qquad\qquad+\Big[\frac{\partial(\mathrm{B}_1 V)}{\partial z_1}-\frac{\partial(\mathrm{B}_2 V)}{\partial z_2}\Big]\delta z\Big\}\cos(N_i, y) \\
&\quad+\Big\{\Big[\frac{\partial(\mathrm{C}_1 V)}{\partial x_1}-\frac{\partial(\mathrm{C}_2 V)}{\partial x_2}\Big]\delta x+\Big[\frac{\partial(\mathrm{C}_1 V)}{\partial y_1}-\frac{\partial(\mathrm{C}_2 V)}{\partial y_2}\Big]\delta y \\
&\qquad\qquad+\Big[\frac{\partial(\mathrm{C}_1 V)}{\partial z_1}-\frac{\partial(\mathrm{C}_2 V)}{\partial z_2}\Big]\delta z\Big\}\cos(N_i, z)\Big]\, dS_{12} \\
&+\mathop{\mathrm{S}}_{S_{12}} V\,[(\mathrm{A}_1-\mathrm{A}_2)\,\delta\cos(N_1, x)+(\mathrm{B}_1-\mathrm{B}_2)\,\delta\cos(N_1, y)+(\mathrm{C}_1-\mathrm{C}_2)\,\delta\cos(N_1, z)]\, dS_{12} \\
&+\mathop{\mathrm{S}}_{S_{12}} V\,[(\mathrm{A}_1-\mathrm{A}_2)\cos(N_1, x)+(\mathrm{B}_1-\mathrm{B}_2)\cos(N_1, y) \\
&\qquad\qquad+(\mathrm{C}_1-\mathrm{C}_2)\cos(N_1, z)]\,\frac{dS'_{12}-dS_{12}}{dS_{12}}\, dS_{12} \\
&=\mathop{\mathrm{S}}_{S_{12}}\Big[\frac{\partial(\mathrm{A}_1 V)}{\partial x_1}-\frac{\partial(\mathrm{A}_2 V)}{\partial x_2}+\frac{\partial(\mathrm{B}_1 V)}{\partial y_1}-\frac{\partial(\mathrm{B}_2 V)}{\partial y_2} \\
&\qquad\qquad+\frac{\partial(\mathrm{C}_1 V)}{\partial z_1}-\frac{\partial(\mathrm{C}_2 V)}{\partial z_2}\Big]\,\delta N_1\, dS_{12}.
\end{aligned}\tag{34}$$

Les égalités (10), (33), (33 bis) et (34) donnent

$$\begin{aligned}
\delta Y=\;&\varepsilon\int_1\Big(\frac{\partial V}{\partial x}\,\delta\mathrm{A}_1+\frac{\partial V}{\partial y}\,\delta\mathrm{B}_1+\frac{\partial V}{\partial z}\,\delta\mathrm{C}_1\Big)\, dv_1 \\
&-\varepsilon\int_1\Big[\frac{\partial V}{\partial x}\Big(\frac{\partial\mathrm{A}_1}{\partial x}\,\delta x+\frac{\partial\mathrm{A}_1}{\partial y}\,\delta y+\frac{\partial\mathrm{A}_1}{\partial z}\,\delta z\Big) \\
&\qquad+\frac{\partial V}{\partial y}\Big(\frac{\partial\mathrm{B}_1}{\partial x}\,\delta x+\frac{\partial\mathrm{B}_1}{\partial y}\,\delta y+\frac{\partial\mathrm{B}_1}{\partial z}\,\delta z\Big) \\
&\qquad+\frac{\partial V}{\partial z}\Big(\frac{\partial\mathrm{C}_1}{\partial x}\,\delta x+\frac{\partial\mathrm{C}_1}{\partial y}\,\delta y+\frac{\partial\mathrm{C}_1}{\partial z}\,\delta z\Big)\Big]\, dv_1 \\
&-\varepsilon\mathop{\mathrm{S}}_{S_1}\Big(\mathrm{A}_1\frac{\partial V}{\partial x_i}+\mathrm{B}_1\frac{\partial V}{\partial y_i}+\mathrm{C}_1\frac{\partial V}{\partial z_i}\Big)\,\delta N_i\, dS_1 \\
&+2\pi\varepsilon\mathop{\mathrm{S}}_{S_1}[\mathrm{A}_1\cos(N_i, x)+\mathrm{B}_1\cos(N_i, y)+\mathrm{C}_1\cos(N_i, z)]^2\,\delta N_i\, dS_1 \\
&+\text{etc.} \\
&-\varepsilon\mathop{\mathrm{S}}_{S_{12}}\Big(\mathrm{A}_1\frac{\partial V}{\partial x_1}+\mathrm{B}_1\frac{\partial V}{\partial y_1}+\mathrm{C}_1\frac{\partial V}{\partial z_1}-\mathrm{A}_2\frac{\partial V}{\partial x_2}-\mathrm{B}_2\frac{\partial V}{\partial y_2}-\mathrm{C}_2\frac{\partial V}{\partial z_2}\Big)\,\delta N_i\, dS_{12} \\
&+2\pi\varepsilon\mathop{\mathrm{S}}_{S_{12}}\{[\mathrm{A}_1\cos(N_1, x)+\mathrm{B}_1\cos(N_1, y)+\mathrm{C}_1\cos(N_1, z)]^2 \\
&\qquad-[\mathrm{A}_2\cos(N_2, x)+\mathrm{B}_2\cos(N_2, y)+\mathrm{C}_2\cos(N_2, z)]^2\}\,\delta N_1\, dS_{12}.
\end{aligned}\tag{35}$$

On pourrait supposer les corps 1 et 2 placés dans le champ engendré par d'autres corps électrisés et polarisés ; *si ces corps sont fixes de forme et de position, si leur état d'électrisation et de polarisation est invariable, enfin s'ils n'ont avec les corps* 1 *et* 2 *aucun point de contact*, l'introduction de ces corps ne modifie pas l'expression de δY; seulement, dans cette expression, V représente alors la *fonction potentielle totale*, provenant non seulement de la polarisation des corps 1 et 2, mais encore de la distribution électrique ou diélectrique répandue sur les corps invariables.

Ajoutons encore une remarque qui nous sera utile au chapître suivant.

La formule précédente, comme toutes celles que nous avons écrites jusqu'ici, a été démontrée en supposant que δx, δy, δz étaient des fonctions continues de x, y, z, admettant des dérivées partielles par rapport à ces variables ; toutefois, il serait aisé de les étendre au cas où les déplacements δx, δy, δz, seraient discontinus le long de certaines surfaces, pourvu que la condition exprimée par l'égalité suivante soit vérifiée en chaque point de ces surfaces :

$$\begin{aligned} &\cos(N, x)\,\delta x + \cos(N, y)\,\delta y + \cos(N, z)\,\delta z \\ &+ \cos(N', x)\,\delta' x + \cos(N', y)\,\delta' y + \cos(N', z)\,\delta' z = 0. \end{aligned} \tag{36}$$

Dans cette égalité, δx, δy, δz, sont les composantes du déplacement du premier côté de la surface ; $\delta' x$, $\delta' y$, $\delta' z$, les composantes du déplacement de l'autre côté ; N est la demi-normale dirigée du premier côté ; N', la demi-normale dirigée du second côté.

L'égalité (35) ne diffère de l'expression de δY dont nous avions fait usage dans nos précédentes publications que par les termes

$$2\pi\varepsilon \int_{S_1} [\mathrm{A}_1 \cos(N_i, x) + \mathrm{B}_1 \cos(N_i, y) + \mathrm{C}_1 \cos(N_i, z)]^2\, \delta N_i\, dS_1,$$

$$2\pi\varepsilon \int_{S_2} [\mathrm{A}_2 \cos(N_i, x) + \mathrm{B}_2 \cos(N_i, y) + \mathrm{C}_2 \cos(N_i, z)]^2\, \delta N_i\, dS_2,$$

$$\begin{aligned} 2\pi\varepsilon \int_{S_{12}} \{ &[\mathrm{A}_1 \cos(N_1, x) + \mathrm{B}_1 \cos(N_1, y) + \mathrm{C}_1 \cos(N_1, z)]^2 \\ &- [\mathrm{A}_2 \cos(N_2, x) + \mathrm{B}_2 \cos(N_2, y) + \mathrm{C}_2 \cos(N_2, z)]^2 \}\, \delta N_1\, dS_{12}. \end{aligned}$$

Ces termes, que nous avions omis, fournissent, dans les applications, les termes complémentaires dont l'introduction a été proposée par M. Liénard.

CHAPÎTRE IV.

Équilibre d'un Fluide incompressible, doué de Force coercitive, et polarisé.

C'est seulement dans le cas où un fluide est incompressible que l'on en peut étudier les conditions d'équilibre mécanique sans avoir besoin de le supposer dénué de force coercitive; dans nos *Leçons sur l'Electricité et le Magnétisme* (Livre IX, Chapître VIII), nous avons donné les conditions d'équilibre d'un pareil fluide; ces conditions sont de deux sortes: les unes doivent être vérifiées en tout point intérieur au fluide; les autres, en tout point de la surface qui le limite; les premières conditions, sous la forme que nous leur avions donnée, étaient exactes; les secondes ne l'étaient pas; M. Liénard a indiqué la valeur du terme que nous avions omis.

Nous allons reprendre ici, en nous appuyant sur l'expression de δY établie au Chapître précédent, l'établissement des conditions d'équilibre d'un fluide polarisé; nous espérons rendre ainsi la démonstration de ces conditions irréprochable au point de vue de la rigueur.

Pour ne pas compliquer outre mesure notre analyse, supposons tout d'abord que la partie déformable du système ne soit formée que d'un seul fluide.

Si nous supposons ce fluide incompressible et si nous négligeons les actions capillaires, la partie variable du potentiel thermodynamique interne du système se réduira à la quantité Y; si l'on désigne par D la densité du fluide et si l'on suppose que les forces auxquelles est soumise chaque masse fluide élémentaire admettent une fonction potentielle Ψ, les forces extérieures appliquées au fluide effectueront, dans toute modification virtuelle, un travail

$$d\mathfrak{T}_e = -\int D\left(\frac{\partial\Psi}{\partial x}\,\delta x + \frac{\partial\Psi}{\partial y}\,\delta y + \frac{\partial\Psi}{\partial z}\,\delta z\right) dv$$
$$+ \mathsf{S} P\,[\cos(P, x)\,\delta x + \cos(P, y)\,\delta y + \cos(P, z)\,\delta z)]\,dS, \tag{1}$$

P désignant la grandeur et la direction de la pression en tout point de l'élément dS, la première intégrale s'étendant au volume entier du fluide, et la seconde à la surface qui le limite.

Nous obtiendrons les conditions d'équilibre du système en écrivant que l'égalité

$$d\mathfrak{T}_e - \delta Y = 0 \tag{2}$$

est vérifiée pour toute modification virtuelle, compatible avec les liaisons, imposée au système.

Nous ne voulons rien supposer sur les lois d'aimantation du fluide; il nous faut donc envisager seulement les modifications dans lesquelles chaque élément de volume, en se déplaçant, entraîne son intensité de polarisation; dès lors, il est aisé de voir que si ω, ω', ω'' désignent les composantes de la rotation élémentaire autour d'axes respectivement parallèles à Ox, Oy, Oz, nous aurons

$$\left.\begin{aligned} \delta \mathrm{A} &= \mathrm{B}\omega'' - \mathrm{C}\omega', \\ \delta \mathrm{B} &= \mathrm{C}\omega - \mathrm{A}\omega'', \\ \delta \mathrm{C} &= \mathrm{A}\omega' - \mathrm{B}\omega. \end{aligned}\right\} \tag{3}$$

Nous savons, d'ailleurs, que

$$\begin{aligned} \omega &= \tfrac{1}{2}\left(\frac{\partial \delta z}{\partial y} - \frac{\partial \delta y}{\partial z}\right), \\ \omega' &= \tfrac{1}{2}\left(\frac{\partial \delta x}{\partial z} - \frac{\partial \delta z}{\partial x}\right), \\ \omega'' &= \tfrac{1}{2}\left(\frac{\partial \delta y}{\partial x} - \frac{\partial \delta x}{\partial y}\right). \end{aligned}$$

Les égalités (3) peuvent donc s'écrire:

$$\left.\begin{aligned} \delta \mathrm{A} &= \tfrac{1}{2}\left[\mathrm{B}\left(\frac{\partial \delta y}{\partial x} - \frac{\partial \delta x}{\partial y}\right) - \mathrm{C}\left(\frac{\partial \delta x}{\partial z} - \frac{\partial \delta z}{\partial x}\right)\right], \\ \delta \mathrm{B} &= \tfrac{1}{2}\left[\mathrm{C}\left(\frac{\partial \delta z}{\partial y} - \frac{\partial \delta y}{\partial z}\right) - \mathrm{A}\left(\frac{\partial \delta y}{\partial x} - \frac{\partial \delta x}{\partial y}\right)\right], \\ \delta \mathrm{C} &= \tfrac{1}{2}\left[\mathrm{A}\left(\frac{\partial \delta x}{\partial z} - \frac{\partial \delta z}{\partial x}\right) - \mathrm{B}\left(\frac{\partial \delta z}{\partial y} - \frac{\partial \delta y}{\partial z}\right)\right]. \end{aligned}\right\} \tag{4}$$

En vertu de ces égalités (4), on a

$$\begin{aligned} &\int\left(\frac{\partial V}{\partial x}\partial \mathrm{A} + \frac{\partial V}{\partial y}\partial \mathrm{B} + \frac{\partial V}{\partial z}\partial \mathrm{C}\right) dv \\ &= \tfrac{1}{2}\int\left\{\frac{\partial V}{\partial x}\left[\mathrm{B}\left(\frac{\partial \delta y}{\partial x} - \frac{\partial \delta x}{\partial y}\right) - \mathrm{C}\left(\frac{\partial \delta x}{\partial z} - \frac{\partial \delta z}{\partial x}\right)\right]\right. \\ &\qquad + \frac{\partial V}{\partial y}\left[\mathrm{C}\left(\frac{\partial \delta z}{\partial y} - \frac{\partial \delta y}{\partial z}\right) - \mathrm{A}\left(\frac{\partial \delta y}{\partial x} - \frac{\partial \delta x}{\partial y}\right)\right] \\ &\qquad \left. + \frac{\partial V}{\partial z}\left[\mathrm{A}\left(\frac{\partial \delta x}{\partial z} - \frac{\partial \delta z}{\partial x}\right) - \mathrm{B}\left(\frac{\partial \delta z}{\partial x} - \frac{\partial \delta y}{\partial z}\right)\right]\right\} dv. \end{aligned} \tag{5}$$

Transformons cette égalité (5) au moyen d'intégrations par parties.

Imaginons que le déplacement dont les composantes sont δx, δy, δz, varie d'une manière continue d'un point à l'autre du fluide, sauf aux divers points d'une surface fermée Σ tracée à l'intérieur du fluide ; soient :

ν la demi-normale à la surface Σ, dirigée vers l'intérieur de cette surface.

ν' la demi-normale à la surface Σ, dirigée vers l'extérieur de cette surface.

δx, δy, δz, les composantes du déplacement à la face interne de la surface Σ.

$\delta' x$, $\delta' y$, $\delta' z$, les composantes du déplacement à la face externe de la surface Σ.

L'égalité précédente peut s'écrire

$$\begin{aligned}
&\int\left(\frac{\partial V}{\partial x}\,\delta \mathrm{A} + \frac{\partial V}{\partial y}\,\delta \mathrm{B} + \frac{\partial V}{\partial z}\,\delta \mathrm{C}\right) dv \\
&= \tfrac{1}{2}\int\left\{\left[\frac{\partial}{\partial y}\left(\mathrm{B}\frac{\partial V}{\partial x} - \mathrm{A}\frac{\partial V}{\partial y}\right) - \frac{\partial}{\partial z}\left(\mathrm{A}\frac{\partial V}{\partial z} - \mathrm{C}\frac{\partial V}{\partial x}\right)\right]\delta x\right. \\
&\qquad + \left[\frac{\partial}{\partial z}\left(\mathrm{C}\frac{\partial V}{\partial y} - \mathrm{B}\frac{\partial V}{\partial z}\right) - \frac{\partial}{\partial x}\left(\mathrm{B}\frac{\partial V}{\partial x} - \mathrm{A}\frac{\partial V}{\partial y}\right)\right]\delta y \\
&\qquad \left. + \left[\frac{\partial}{\partial x}\left(\mathrm{A}\frac{\partial V}{\partial z} - \mathrm{C}\frac{\partial V}{\partial x}\right) - \frac{\partial}{\partial y}\left(\mathrm{C}\frac{\partial V}{\partial y} - \mathrm{B}\frac{\partial V}{\partial z}\right)\right]\delta z\right\} dv \\
&+ \tfrac{1}{2}\mathrm{S}_{S}\left\{\left[\left(\mathrm{B}\frac{\partial V}{\partial x_i} - \mathrm{A}\frac{\partial V}{\partial y_i}\right)\cos(N_i, y) - \left(\mathrm{A}\frac{\partial V}{\partial z_i} - \mathrm{C}\frac{\partial V}{\partial x_i}\right)\cos(N_i, z)\right]\delta x\right. \\
&\qquad + \left[\left(\mathrm{C}\frac{\partial V}{\partial y_i} - \mathrm{B}\frac{\partial V}{\partial z_i}\right)\cos(N_i, z) - \left(\mathrm{B}\frac{\partial V}{\partial x_i} - \mathrm{A}\frac{\partial V}{\partial y_i}\right)\cos(N_i, x)\right]\delta y \\
&\qquad \left. + \left[\left(\mathrm{A}\frac{\partial V}{\partial z_i} - \mathrm{C}\frac{\partial V}{\partial x_i}\right)\cos(N_i, x) - \left(\mathrm{C}\frac{\partial V}{\partial y_i} - \mathrm{B}\frac{\partial V}{\partial z_i}\right)\cos(N_i, y)\right]\delta z\right\} dS \\
&+ \tfrac{1}{2}\mathrm{S}_{\Sigma}\left\{\left[\left(\mathrm{B}\frac{\partial V}{\partial x} - \mathrm{A}\frac{\partial V}{\partial y}\right)\cos(\nu, y) - \left(\mathrm{A}\frac{\partial V}{\partial z} - \mathrm{C}\frac{\partial V}{\partial x}\right)\cos(\nu, z)\right](\delta x - \delta' x)\right. \\
&\qquad + \left[\left(\mathrm{C}\frac{\partial V}{\partial y} - \mathrm{B}\frac{\partial V}{\partial z}\right)\cos(\nu, z) - \left(\mathrm{B}\frac{\partial V}{\partial x} - \mathrm{A}\frac{\partial V}{\partial y}\right)\cos(\nu, x)\right](\delta y - \delta' y) \\
&\qquad + \left[\left(\mathrm{A}\frac{\partial V}{\partial z} - \mathrm{C}\frac{\partial V}{\partial x}\right)\cos(\nu, x)\right. \\
&\qquad\qquad \left.\left. - \left(\mathrm{C}\frac{\partial V}{\partial y} - \mathrm{B}\frac{\partial V}{\partial z}\right)\cos(\nu, y)\right](\delta z - \delta' z)\right\} d\Sigma. \qquad (6)
\end{aligned}$$

L'égalité (35) du Chapitre précédent, jointe aux égalités (1) et (6) du présent Chapitre, permet d'écrire explicitement la condition d'équilibre (2).

Cette condition (2) ne doit pas avoir lieu quels que soient δx, δy, δz; ces quantités sont assujetties à deux conditions:

1°. En tout point du fluide, que l'on suppose incompressible, on doit avoir:

$$\frac{\partial \delta x}{\partial x} + \frac{\partial \delta y}{\partial y} + \frac{\partial \delta z}{\partial z} = 0. \tag{7}$$

2°. En tout point de la surface Σ, on doit avoir [Chapître III, condition (36)]:

$$\cos(\nu, x)(\delta x - \delta' x) + \cos(\nu, y)(\delta y - \delta' y) + \cos(\nu, z)(\delta z - \delta' z) = 0. \tag{8}$$

Dès lors, il doit exister:

1°. Une quantité Π, fonction continue d'x, y, z, dans toute l'étendue du fluide;

2°. Une quantité λ variable d'une manière continue sur la surface Σ;

telles que l'on ait *identiquement*, quels que soient δx, δy, δz,

$$\begin{aligned} & d\mathfrak{T}_e - \delta Y \\ & + \int \Pi \left(\frac{\partial \delta x}{\partial x} + \frac{\partial \delta y}{\partial y} + \frac{\partial \delta z}{\partial z} \right) dv \\ & + \mathop{\mathrm{S}}_{\Sigma} \lambda \left[\cos(\nu, x)(\delta x - \delta' x) + \cos(\nu, y)(\delta y - \delta' y) + \cos(\nu, z)(\delta z - \delta' z) \right] d\Sigma = 0. \end{aligned}$$

Des intégrations par parties permettent de transformer cette identité en

$$\begin{aligned} & d\mathfrak{T}_e - \delta Y \\ & - \int \left(\frac{\partial \Pi}{\partial x} \delta x + \frac{\partial \Pi}{\partial y} \delta y + \frac{\partial \Pi}{\partial z} \delta z \right) dv \\ & - \mathop{\mathrm{S}}_{S} \Pi \left[\cos(N_i, x) \delta x + \cos(N_i, y) \delta y + \cos(N_i, z) \delta z \right] dS \\ & + \mathop{\mathrm{S}}_{\Sigma} (\lambda - \Pi) [\cos(\nu, x)(\delta x - \delta' x) + \cos(\nu, y)(\delta y - \delta' y) \\ & \qquad + \cos(\nu, z)(\delta z - \delta' z)] \, d\Sigma = 0. \end{aligned} \tag{9}$$

L'égalité (35) du Chapître précédent, jointe aux égalités (1), (6) et (9) du présent Chapître donnent:

$$
\begin{aligned}
\int \biggl[\biggl\{ & \frac{\partial \Pi}{\partial x} + D \frac{\partial \Psi}{\partial x} - \varepsilon \left(\frac{\partial V}{\partial x} \frac{\partial \mathrm{A}}{\partial x} + \frac{\partial V}{\partial y} \frac{\partial \mathrm{B}}{\partial x} + \frac{\partial V}{\partial z} \frac{\partial \mathrm{C}}{\partial x} \right) \\
& + \frac{\varepsilon}{2} \biggl[\frac{\partial}{\partial y} \left(\mathrm{B} \frac{\partial V}{\partial x} - \mathrm{A} \frac{\partial V}{\partial y} \right) - \frac{\partial}{\partial z} \left(\mathrm{A} \frac{\partial V}{\partial z} - \mathrm{C} \frac{\partial V}{\partial x} \right) \biggr] \biggr\} \delta x \\
+ \biggl\{ & \frac{\partial \Pi}{\partial y} + D \frac{\partial \Psi}{\partial y} - \varepsilon \left(\frac{\partial V}{\partial x} \frac{\partial \mathrm{A}}{\partial y} + \frac{\partial V}{\partial y} \frac{\partial \mathrm{B}}{\partial y} + \frac{\partial V}{\partial z} \frac{\partial \mathrm{C}}{\partial y} \right) \\
& + \frac{\varepsilon}{2} \biggl[\frac{\partial}{\partial z} \left(\mathrm{C} \frac{\partial V}{\partial y} - \mathrm{B} \frac{\partial V}{\partial z} \right) - \frac{\partial}{\partial x} \left(\mathrm{B} \frac{\partial V}{\partial x} - \mathrm{A} \frac{\partial V}{\partial y} \right) \biggr] \biggr\} \delta y \\
+ \biggl\{ & \frac{\partial \Pi}{\partial z} + D \frac{\partial \Psi}{\partial z} - \varepsilon \left(\frac{\partial V}{\partial x} \frac{\partial \mathrm{A}}{\partial z} + \frac{\partial V}{\partial y} \frac{\partial \mathrm{B}}{\partial z} + \frac{\partial V}{\partial z} \frac{\partial \mathrm{C}}{\partial z} \right) \\
& + \frac{\varepsilon}{2} \biggl[\frac{\partial}{\partial x} \left(\mathrm{A} \frac{\partial V}{\partial z} - \mathrm{C} \frac{\partial V}{\partial x} \right) - \frac{\partial}{\partial y} \left(\mathrm{C} \frac{\partial V}{\partial y} - \mathrm{B} \frac{\partial V}{\partial z} \right) \biggr] \biggr\} \delta z \biggr] dv \\
+ \mathrm{S}_{\Sigma} \biggl[\biggl\{ & (\Pi - \lambda) \cos(\nu, x) + \frac{\varepsilon}{2} \biggl[\left(\mathrm{B} \frac{\partial V}{\partial x} - \mathrm{A} \frac{\partial V}{\partial y} \right) \cos(\nu, y) \\
& \qquad - \left(\mathrm{A} \frac{\partial V}{\partial z} - \mathrm{C} \frac{\partial V}{\partial x} \right) \cos(\nu, z) \biggr] \biggr\} (\delta x - \delta' x) \\
+ \biggl\{ & (\Pi - \lambda) \cos(\nu, y) + \frac{\varepsilon}{2} \biggl[\left(\mathrm{C} \frac{\partial V}{\partial y} - \mathrm{B} \frac{\partial V}{\partial z} \right) \cos(\nu, z) \\
& \qquad - \left(\mathrm{B} \frac{\partial V}{\partial x} - \mathrm{A} \frac{\partial V}{\partial y} \right) \cos(\nu, x) \biggr] \biggr\} (\delta y - \delta' y) \\
+ \biggl\{ & (\Pi - \lambda) \cos(\nu, z) + \frac{\varepsilon}{2} \biggl[\left(\mathrm{A} \frac{\partial V}{\partial z} - \mathrm{C} \frac{\partial V}{\partial x} \right) \cos(\nu, x) \\
& \qquad - \left(\mathrm{C} \frac{\partial V}{\partial y} - \mathrm{B} \frac{\partial V}{\partial z} \right) \cos(\nu, y) \biggr] \biggr\} (\delta z - \delta' z) \biggr] d\Sigma \\
+ \mathrm{S}_{s} \biggl[\biggl\{ & \biggl[\Pi - \varepsilon \left(\mathrm{A} \frac{\partial V}{\partial x_i} + \mathrm{B} \frac{\partial V}{\partial y_i} + \mathrm{C} \frac{\partial V}{\partial z_i} \right) \\
& + 2\pi\varepsilon \left(\mathrm{A} \cos(N_i, x) + \mathrm{B} \cos(N_i, y) + \mathrm{C} \cos(N_i, z) \right)^2 \biggr] \cos(N_i, x) \\
& + \frac{\varepsilon}{2} \biggl[\left(\mathrm{B} \frac{\partial V}{\partial x_i} - \mathrm{A} \frac{\partial V}{\partial y_i} \right) \cos(N_i, y) - \left(\mathrm{A} \frac{\partial V}{\partial z_i} - \mathrm{C} \frac{\partial V}{\partial x_i} \right) \cos(N_i, z) \biggr] \\
& \qquad - P \cos(P, x) \biggr\} \delta x \\
+ \biggl\{ & \biggl[\Pi - \varepsilon \left(\mathrm{A} \frac{\partial V}{\partial x_i} + \mathrm{B} \frac{\partial V}{\partial y_i} + \mathrm{C} \frac{\partial V}{\partial z_i} \right) \\
& + 2\pi\varepsilon \left(\mathrm{A} \cos(N_i, x) + \mathrm{B} \cos(N_i, y) + \mathrm{C} \cos(N_i, z) \right)^2 \biggr] \cos(N_i, y) \\
& + \frac{\varepsilon}{2} \biggl[\left(\mathrm{C} \frac{\partial V}{\partial y_i} - \mathrm{B} \frac{\partial V}{\partial z_i} \right) \cos(N_i, z) - \left(\mathrm{B} \frac{\partial V}{\partial x_i} - \mathrm{A} \frac{\partial V}{\partial y_i} \right) \cos(N_i, x) \biggr] \\
& \qquad - P \cos(P, y) \biggr\} \delta y \\
+ \biggl\{ & \biggl[\Pi - \varepsilon \left(\mathrm{A} \frac{\partial V}{\partial x_i} + \mathrm{B} \frac{\partial V}{\partial y_i} + \mathrm{C} \frac{\partial V}{\partial z_i} \right) \\
& + 2\pi\varepsilon \left(\mathrm{A} \cos(N_i, x) + \mathrm{B} \cos(N_i, y) + \mathrm{C} \cos(N_i, z) \right)^2 \biggr] \cos(N_i, z) \\
& + \frac{\varepsilon}{2} \biggl[\left(\mathrm{A} \frac{\partial V}{\partial z_i} - \mathrm{C} \frac{\partial V}{\partial x_i} \right) \cos(N_i, x) - \left(\mathrm{C} \frac{\partial V}{\partial y_i} - \mathrm{B} \frac{\partial V}{\partial z_i} \right) \cos(N_i, y) \biggr] \\
& \qquad - P \cos(P, z) \biggr\} \delta z \biggr] d\Sigma = 0. \qquad (10)
\end{aligned}
$$

Cette égalité doit avoir lieu quels que soient δx, δy, δz; on en conclut sans peine que l'on doit avoir:

1°. En tout point intérieur au fluide:

$$\left.\begin{aligned}&\frac{\partial \Pi}{\partial x} + D\frac{\partial \Psi}{\partial x} - \varepsilon\left(\frac{\partial V}{\partial x}\frac{\partial \mathrm{A}}{\partial x} + \frac{\partial V}{\partial y}\frac{\partial \mathrm{B}}{\partial x} + \frac{\partial V}{\partial z}\frac{\partial \mathrm{C}}{\partial x}\right)\\ &\quad + \frac{\varepsilon}{2}\left[\frac{\partial}{\partial y}\left(\mathrm{B}\frac{\partial V}{\partial x} - \mathrm{A}\frac{\partial V}{\partial y}\right) - \frac{\partial}{\partial z}\left(\mathrm{A}\frac{\partial V}{\partial z} - \mathrm{C}\frac{\partial V}{\partial x}\right)\right] = 0,\end{aligned}\right\} \tag{11}$$

et deux autres égalités analogues.

2°. En tout point de la surface qui limite le fluide:

$$\left.\begin{aligned}&\Big[\Pi - \varepsilon\left(\mathrm{A}\frac{\partial V}{\partial x_i} + \mathrm{B}\frac{\partial V}{\partial y_i} + \mathrm{C}\frac{\partial V}{\partial z_i}\right)\\ &\qquad + 2\pi\varepsilon\left(\mathrm{A}\cos(N_i, x) + \mathrm{B}\cos(N_i, y) + \mathrm{C}\cos(N_i, z)\right)^2\Big]\cos(N_i, x)\\ &+ \frac{\varepsilon}{2}\Big[\left(\mathrm{B}\frac{\partial V}{\partial x_i} - \mathrm{A}\frac{\partial V}{\partial y_i}\right)\cos(N_i, y)\\ &\qquad\qquad - \left(\mathrm{A}\frac{\partial V}{\partial z_i} - \mathrm{C}\frac{\partial V}{\partial x_i}\right)\cos(N_i, z)\Big] = P\cos(P, x),\end{aligned}\right\} \tag{12}$$

et deux autres égalités analogues.

3°. En tout point de la surface Σ:

$$\left.\begin{aligned}&(\Pi - \lambda)\cos(\nu, x)\\ &\quad + \frac{\varepsilon}{2}\left[\left(\mathrm{B}\frac{\partial V}{\partial x} - \mathrm{A}\frac{\partial V}{\partial y}\right)\cos(\nu, y) - \left(\mathrm{A}\frac{\partial V}{\partial z} - \mathrm{C}\frac{\partial V}{\partial x}\right)\cos(\nu, z)\right] = 0,\\ &(\Pi - \lambda)\cos(\nu, y)\\ &\quad + \frac{\varepsilon}{2}\left[\left(\mathrm{C}\frac{\partial V}{\partial y} - \mathrm{B}\frac{\partial V}{\partial z}\right)\cos(\nu, z) - \left(\mathrm{B}\frac{\partial V}{\partial x} - \mathrm{A}\frac{\partial V}{\partial y}\right)\cos(\nu, x)\right] = 0,\\ &(\Pi - \lambda)\cos(\nu, z)\\ &\quad + \frac{\varepsilon}{2}\left[\left(\mathrm{A}\frac{\partial V}{\partial z} - \mathrm{C}\frac{\partial V}{\partial x}\right)\cos(\nu, x) - \left(\mathrm{C}\frac{\partial V}{\partial y} - \mathrm{B}\frac{\partial V}{\partial z}\right)\cos(\nu, y)\right] = 0.\end{aligned}\right\} \tag{13}$$

Multiplions la première des égalités (13) par $\cos(\nu, x)$, la seconde par $\cos(\nu, y)$, la troisième par $\cos(\nu, z)$; ajoutons membre à membre les résultats obtenus; nous trouvons l'égalité

$$\Pi - \lambda = 0,$$

moyennant laquelle les égalités (13) deviennent:

$$\left.\begin{aligned}
&\left(\mathrm{B}\frac{\partial V}{\partial x}-\mathrm{A}\frac{\partial V}{\partial y}\right)\cos(\nu,y)-\left(\mathrm{A}\frac{\partial V}{\partial z}-\mathrm{C}\frac{\partial V}{\partial x}\right)\cos(\nu,z)=0,\\
&\left(\mathrm{C}\frac{\partial V}{\partial y}-\mathrm{B}\frac{\partial V}{\partial z}\right)\cos(\nu,z)-\left(\mathrm{B}\frac{\partial V}{\partial x}-\mathrm{A}\frac{\partial V}{\partial y}\right)\cos(\nu,x)=0,\\
&\left(\mathrm{A}\frac{\partial V}{\partial z}-\mathrm{C}\frac{\partial V}{\partial x}\right)\cos(\nu,x)-\left(\mathrm{C}\frac{\partial V}{\partial y}-\mathrm{B}\frac{\partial V}{\partial z}\right)\cos(\nu,y)=0.
\end{aligned}\right\}\quad(14)$$

Remarquons maintenant que la surface Σ est entièrement arbitraire; quel que soit le point du fluide que l'on considère et quelle que soit la demi-droite ν issue de ce point, on pourra faire passer la surface Σ par ce point, et cela de telle sorte qu'elle soit normale à la droite ν; les égalités (14) doivent donc être vérifiées en tout point du fluide, et quelle que soit la direction ν; *on doit donc avoir, en tout point du fluide,*

$$\left.\begin{aligned}
&\mathrm{C}\frac{\partial V}{\partial y}-\mathrm{B}\frac{\partial V}{\partial z}=0,\\
&\mathrm{A}\frac{\partial V}{\partial z}-\mathrm{C}\frac{\partial V}{\partial x}=0,\\
&\mathrm{B}\frac{\partial V}{\partial x}-\mathrm{A}\frac{\partial V}{\partial y}=0.
\end{aligned}\right\}\quad(15)$$

Ces égalités, rapprochées des égalités (11), montrent que *l'on doit aussi avoir, en tout point du fluide:*

$$\left.\begin{aligned}
&\frac{\partial\Pi}{\partial x}+D\frac{\partial\Psi}{\partial x}-\varepsilon\left(\frac{\partial V}{\partial x}\frac{\partial\mathrm{A}}{\partial x}+\frac{\partial V}{\partial y}\frac{\partial\mathrm{B}}{\partial x}+\frac{\partial V}{\partial z}\frac{\partial\mathrm{C}}{\partial x}\right)=0,\\
&\frac{\partial\Pi}{\partial y}+D\frac{\partial\Psi}{\partial y}-\varepsilon\left(\frac{\partial V}{\partial x}\frac{\partial\mathrm{A}}{\partial y}+\frac{\partial V}{\partial y}\frac{\partial\mathrm{B}}{\partial y}+\frac{\partial V}{\partial z}\frac{\partial\mathrm{C}}{\partial y}\right)=0,\\
&\frac{\partial\Pi}{\partial z}+D\frac{\partial\Psi}{\partial z}-\varepsilon\left(\frac{\partial V}{\partial x}\frac{\partial\mathrm{A}}{\partial z}+\frac{\partial V}{\partial y}\frac{\partial\mathrm{B}}{\partial z}+\frac{\partial V}{\partial z}\frac{\partial\mathrm{C}}{\partial z}\right)=0.
\end{aligned}\right\}\quad(16)$$

Les égalités (15), rapprochées des égalités (12), montrent que *l'on doit avoir, en tout point de la surface qui limite le fluide:*

$$\left.\begin{aligned}
P\cos(P,x)=\Big[&\Pi-\varepsilon\left(\mathrm{A}\frac{\partial V}{\partial x_i}+\mathrm{B}\frac{\partial V}{\partial y_i}+\mathrm{C}\frac{\partial V}{\partial z_i}\right)\\
&+2\pi\varepsilon\left(\mathrm{A}\cos(N_i,x)+\mathrm{B}\cos(N_i,y)+\mathrm{C}\cos(N_i,z)\right)^2\Big]\cos(N_i,x),\\
P\cos(P,y)=\Big[&\Pi-\varepsilon\left(\mathrm{A}\frac{\partial V}{\partial x_i}+\mathrm{B}\frac{\partial V}{\partial y_i}+\mathrm{C}\frac{\partial V}{\partial z_i}\right)\\
&+2\pi\varepsilon\left(\mathrm{A}\cos(N_i,x)+\mathrm{B}\cos(N_i,y)+\mathrm{C}\cos(N_i,z)\right)^2\Big]\cos(N_i,y),\\
P\cos(P,z)=\Big[&\Pi-\varepsilon\left(\mathrm{A}\frac{\partial V}{\partial x_i}+\mathrm{B}\frac{\partial V}{\partial y_i}+\mathrm{C}\frac{\partial V}{\partial z_i}\right)\\
&+2\pi\varepsilon\left(\mathrm{A}\cos(N_i,x)+\mathrm{B}\cos(N_i,y)+\mathrm{C}\cos(N_i,z)\right)^2\Big]\cos(N_i,z).
\end{aligned}\right\}\quad(17)$$

Les égalités (15), (16), (17) représentent les *conditions d'équilibre mécanique d'un fluide incompressible polarisé, doué ou non de force coercitive.*

Interprétons ces conditions.

Les égalités (17) nous apprennent que, pour maintenir le fluide en équilibre, il faut appliquer en chaque point de la surface qui le termine, une pression *normale* à cette surface; la grandeur de cette pression, positive pour une pression dirigée vers l'intérieur du fluide, est représentée par l'égalité:

$$P = \Pi - \varepsilon\left(A\frac{\partial V}{\partial x_i} + B\frac{\partial V}{\partial y_i} + C\frac{\partial V}{\partial z_i}\right) + 2\pi\varepsilon\left(A\cos(N_i, x) + B\cos(N_i, y) + C\cos(N_i, z)\right)^2. \tag{18}$$

On voit que *la grandeur de cette pression dépend de l'orientation de l'élément sur lequel elle agit;* ce résultat remarquable est dû à M. Liénard; le terme

$$2\pi\varepsilon\left(A\cos(N_i, x) + B\cos(N_i, y) + C\cos(N_i, z)\right)^2 = 2\pi\varepsilon M^2\cos^2(M, N_i)$$

avait été omis dans l'expression de P qui est donnée dans nos *Leçons sur l'Electricité et le Magnétisme.*

En vertu des égalités (12), il doit exister une fonction $\theta(x, y, z)$ telle que l'on ait, en tout point du fluide,

$$\left.\begin{aligned} A &= -\varepsilon\theta\frac{\partial V}{\partial x}, \\ B &= -\varepsilon\theta\frac{\partial V}{\partial y}, \\ C &= -\varepsilon\theta\frac{\partial V}{\partial z}. \end{aligned}\right\} \tag{19}$$

Si le fluide considéré est homogène, cas auquel D est indépendant d'x, y, z, les égalités (16) donnent, moyennant les égalités (19),

$$d(\Pi + D\Phi) + \frac{1}{\theta}(A\,dA + B\,dB + C\,dC) = 0,$$

ou bien, à cause de l'égalité

$$A\,dA + B\,dB + C\,dC = M\,dM,$$

$$d(\Pi + D\Phi) + \frac{M}{\theta}\,dM = 0.$$

La quantité $\frac{M}{\theta}\,dM$ ne pourrait être une différentielle totale, si θ dépendait de x, y, z, autrement que par l'intermédiaire de la variable M. Il existe donc une fonction $\Theta(\mathrm{M})$ telle que l'on ait, en chaque point du fluide

$$\theta(x, y, z) = \Theta(\mathrm{M}),$$

en sorte que les égalités (19) peuvent s'écrire:

$$\left.\begin{aligned} \mathrm{A} &= -\varepsilon\Theta(\mathrm{M})\frac{\partial V}{\partial x}, \\ \mathrm{B} &= -\varepsilon\Theta(\mathrm{M})\frac{\partial V}{\partial y}, \\ \mathrm{C} &= -\varepsilon\Theta(\mathrm{M})\frac{\partial V}{\partial z}. \end{aligned}\right\} \tag{20}$$

Ainsi, *lorsqu'un fluide, même doué de force coercitive, est en équilibre, la polarisation y est distribuée comme elle le serait sur un fluide parfaitement doux, de même forme, dont le coefficient de polarisation serait une fonction, convenablement choisie, de l'intensité de polarisation; le choix de cette fonction* $\Theta(\mathrm{M})$ *ne dépend pas seulement de la nature du fluide étudié; il peut dépendre de la suite des modifications qui ont amené le fluide à l'état d'équilibre.*

C'est le résultat fondamental que nous avions obtenu, dans nos *Leçons sur l'Electricité et le Magnétisme*, par une analyse moins générale et moins rigoureuse.

Tout ce qui précède suppose le système réduit à un fluide unique.

Imaginons maintenant qu'il soit formé de deux fluides en contact, 1 et 2. Pour chacun de ces deux fluides, on aura à écrire des égalités analogues aux égalités (15), (16) et (17), en affectant les quantités qui y figurent de l'indice 1 pour le premier fluide et de l'indice 2 pour le second; en outre, nous devrons avoir, en tout point de la surface de contact S_{12} des deux fluides:

$$\begin{aligned} \Pi_1 - \Pi_2 = \varepsilon\Big[&\Big(\mathrm{A}_1\frac{\partial V}{\partial x_1} + \mathrm{B}_1\frac{\partial V}{\partial y_1} + \mathrm{C}_1\frac{\partial V}{\partial z_1}\Big) - \Big(\mathrm{A}_2\frac{\partial V}{\partial x_2} + \mathrm{B}_2\frac{\partial V}{\partial y_2} + \mathrm{C}_2\frac{\partial V}{\partial z_2}\Big)\Big] \\ &- 2\pi\varepsilon\,[(\mathrm{A}_1\cos(N_1, x) + \mathrm{B}_1\cos(N_1, y) + \mathrm{C}_1\cos(N_1, z))^2 \\ &\qquad - (\mathrm{A}_2\cos(N_2, x) + \mathrm{B}_2\cos(N_2, y) + \mathrm{C}_2\cos(N_2, z))^2]. \end{aligned}$$

L'établissement de cette condition ne présente aucune difficulté.

CHAPÎTRE V.

Les Fluides parfaitement doux.

Prenons un système formé de deux fluides, 1 et 2, dénués de force coercitive et compressibles. Si nous négligeons les actions capillaires, le Potentiel Thermodynamique Interne de ce système pourra se mettre sous la forme

$$F = \int_1 \Phi_1(D_1)\, dv_1 + \int_2 \Phi_2(D_2)\, dv_2 + Y + \int_1 F_1(M_1, D_1)\, dv_1 + \int_2 F_2(M_2, D_2)\, dv_2. \quad (1)$$

Quant au travail virtuel $d\mathfrak{T}_e$ des forces extérieures, il sera donné comme au chapître précédent, par l'expression

$$\begin{aligned} d\mathfrak{T}_e = & -\int_1 D_1 \left(\frac{\partial \Psi}{\partial x}\, \delta x + \frac{\partial \Psi}{\partial y}\, \delta y + \frac{\partial \Psi}{\partial z}\, \delta z\right) dv_1 \\ & -\int_2 D_2 \left(\frac{\partial \Psi}{\partial x}\, \delta x + \frac{\partial \Psi}{\partial y}\, \delta y + \frac{\partial \Psi}{\partial z}\, \delta z\right) dv_2 \\ & +\mathop{\mathrm{S}}_{S_1} P\, [\cos(P, x)\, \delta x + \cos(P, y)\, \delta y + \cos(P, z)\, \delta z]\, dS_1 \\ & +\mathop{\mathrm{S}}_{S_2} P\, [\cos(P, x)\, \delta x + \cos(P, y)\, \delta y + \cos(P, z)\, \delta z]\, dS_2. \end{aligned} \quad (2)$$

Les conditions d'équilibre du système s'obtiendront en exprimant que l'égalité

$$d\mathfrak{T}_e - \delta F = 0 \quad (3)$$

est vérifiée pour toutes les modifications virtuelles du système.

Il nous est loisible de considérer d'abord les seules modifications dans lesquelles chaque élément matériel garde un volume invariable et entraîne avec lui sa polarisation ; l'égalité (3), appliquée à de semblables modifications, donne les égalités (15), (16), (17), et (21) du Chapître précédent à titre de *conditions nécessaires, mais non suffisantes, de l'équilibre.*

Ecrivons maintenant l'expression générale de $d\mathfrak{T}_e - \delta F$.

Nous avons

$$\left.\begin{aligned} \delta D_1 &= -D_1\left(\frac{\partial\delta x}{\partial x}+\frac{\partial\delta y}{\partial y}+\frac{\partial\delta z}{\partial z}\right), \\ \delta D_2 &= -D_2\left(\frac{\partial\delta x}{\partial y}+\frac{\partial\delta y}{\partial y}+\frac{\partial\delta z}{\partial z}\right), \end{aligned}\right\} \tag{4}$$

$$\left.\begin{aligned} \delta \mathrm{M}_1 &= \frac{\mathrm{A}_1}{\mathrm{M}_1}\,\delta\mathrm{A}_1+\frac{\mathrm{B}_1}{\mathrm{M}_1}\,\delta\mathrm{B}_1+\frac{\mathrm{C}_1}{\mathrm{M}_1}\,\delta\mathrm{C}_1, \\ \delta \mathrm{M}_2 &= \frac{\mathrm{A}_2}{\mathrm{M}_2}\,\delta\mathrm{A}_2+\frac{\mathrm{B}_2}{\mathrm{M}_2}\,\delta\mathrm{B}_2+\frac{\mathrm{C}_2}{\mathrm{M}_2}\,\delta\mathrm{C}_2. \end{aligned}\right\} \tag{5}$$

Si donc nous définissons les deux fonctions $f_1(\mathrm{M}_1, D_1)$, $f_2(\mathrm{M}_2, D_2)$ par les égalités

$$\left.\begin{aligned} \frac{1}{f_1(\mathrm{M}_1, D_1)} &= \frac{1}{\mathrm{M}_1}\,\frac{\partial \mathrm{F}_1(\mathrm{M}_1, D_1)}{\partial D_1}, \\ \frac{1}{f_2(\mathrm{M}_2, D_2)} &= \frac{1}{\mathrm{M}_2}\,\frac{\partial \mathrm{F}_2(\mathrm{M}_2, D_2)}{\partial D_2}, \end{aligned}\right\} \tag{6}$$

nous aurons, en général,

$$\begin{aligned} \delta\mathrm{F} = \delta Y &+ \int_1 \frac{1}{f_1(\mathrm{M}_1, D_1)}\,(\mathrm{A}_1\delta\mathrm{A}_1+\mathrm{B}_1\delta\mathrm{B}_1+\mathrm{C}_1\delta\mathrm{C}_1)\,dv_1 \\ &+ \int_2 \frac{1}{f_2(\mathrm{M}_2, D_2)}\,(\mathrm{A}_2\delta\mathrm{A}_2+\mathrm{B}_2\delta\mathrm{B}_2+\mathrm{C}_2\delta\mathrm{C}_2)\,dv_2 \\ &+ \int_1 \left[\phi_1(D_1) - D_1\frac{d\phi_1(D_1)}{dD_1} + f_1(\mathrm{M}_1, D_1) - D_1\frac{\partial f_1(\mathrm{M}_1, D_1)}{\partial D_1}\right] \\ &\qquad\qquad \times\left(\frac{\partial\delta x}{\partial x}+\frac{\partial\delta y}{\partial y}+\frac{\partial\delta z}{\partial z}\right)dv_1 \\ &+ \int_2 \left[\phi_2(D_2) - D_2\frac{d\phi_2(D_2)}{dD_2} + f_2(\mathrm{M}_2, D_2) - D_2\frac{\partial f_2(\mathrm{M}_2, D_2)}{\partial D_2}\right] \\ &\qquad\qquad \times\left(\frac{\partial\delta x}{\partial x}+\frac{\partial\delta y}{\partial y}+\frac{\partial\delta z}{\partial z}\right)dv_2. \end{aligned} \tag{7}$$

Observons que l'on a identiquement

$$\left.\begin{aligned} \mathrm{A}_1(\mathrm{B}_1\omega''-\mathrm{C}_1\omega')+\mathrm{B}_1(\mathrm{C}_1\omega-\mathrm{A}_1\omega'')+\mathrm{C}_1(\mathrm{A}_1\omega'-\mathrm{B}_1\omega) &= 0, \\ \mathrm{A}_2(\mathrm{B}_2\omega''-\mathrm{C}_2\omega')+\mathrm{B}_2(\mathrm{C}_2\omega-\mathrm{A}_2\omega'')+\mathrm{C}_2(\mathrm{A}_2\omega''-\mathrm{B}_2\omega) &= 0, \end{aligned}\right\} \tag{8}$$

et nous verrons sans peine que les égalités (35) du Chapître III, (2) et (7) du présent Chapître, permettent d'écrire l'égalité (3) sous la forme :

$$
\begin{aligned}
&\int_1 \Big\{ \Big[\varepsilon \frac{\partial V}{\partial x} + \frac{\mathrm{A}_1}{f_1(\mathrm{M}_1, D_1)} \Big] \Delta \mathrm{A}_1 + \Big[\varepsilon \frac{\partial V}{\partial y} + \frac{\mathrm{B}_1}{f_1(\mathrm{M}_1, D_1)} \Big] \Delta \mathrm{B}_1 \\
&\qquad\qquad + \Big[\varepsilon \frac{\partial V}{\partial z} + \frac{\mathrm{C}_1}{f_1(\mathrm{M}_1, D_1)} \Big] \Delta \mathrm{C}_1 \Big\} dv_1 \\
&+ \int_1 \Big[\phi_1(D_1) - D_1 \frac{d\phi_1(D_1)}{dD_1} + f_1(\mathrm{M}_1, D_1) - D_1 \frac{\partial f_1(\mathrm{M}_1, D_1)}{\partial D_1} \Big] \\
&+ \text{etc.} \qquad\qquad\qquad\qquad \times \Big(\frac{\partial \delta x}{\partial x} + \frac{\partial \delta y}{\partial y} + \frac{\partial \delta z}{\partial z} \Big) dv_1 \\
&+ \varepsilon \int_1 \Big[\frac{\partial V}{\partial x} (\mathrm{B}_1 \omega'' - \mathrm{C}_1 \omega') + \frac{\partial V}{\partial y} (\mathrm{C}_1 \omega - \mathrm{A}_1 \omega'') + \frac{\partial V}{\partial z} (\mathrm{A}_1 \omega' - \mathrm{B}_1 \omega) \Big] dv_1 \\
&+ \int_1 \Big\{ \Big[D_1 \frac{\partial \Psi}{\partial x} - \varepsilon \Big(\frac{\partial V}{\partial x} \frac{\partial \mathrm{A}_1}{\partial x} + \frac{\partial V}{\partial y} \frac{\partial \mathrm{B}_1}{\partial x} + \frac{\partial V}{\partial z} \frac{\partial \mathrm{C}_1}{\partial x} \Big) \Big] \delta x \\
&\quad + \Big[D_1 \frac{\partial \Psi}{\partial y} - \varepsilon \Big(\frac{\partial V}{\partial x} \frac{\partial \mathrm{A}_1}{\partial y} + \frac{\partial V}{\partial y} \frac{\partial \mathrm{B}_1}{\partial y} + \frac{\partial V}{\partial z} \frac{\partial \mathrm{C}_1}{\partial y} \Big) \Big] \delta y \\
&\quad + \Big[D_1 \frac{\partial \Psi}{\partial z} - \varepsilon \Big(\frac{\partial V}{\partial x} \frac{\partial \mathrm{A}_1}{\partial z} + \frac{\partial V}{\partial y} \frac{\partial \mathrm{B}_1}{\partial z} + \frac{\partial V}{\partial z} \frac{\partial \mathrm{C}_1}{\partial z} \Big) \Big] \delta z \Big\} dv_1 \\
&+ \mathrm{S}_{S_1} \Big\{ \Big[2\pi\varepsilon \big(\mathrm{A}_1 \cos(N_i, x) + \mathrm{B}_1 \cos(N_i, y) + \mathrm{C}_1 \cos(N_i, z) \big)^2 \\
&\qquad - \varepsilon \Big(\mathrm{A}_1 \frac{\partial V}{\partial x_i} + \mathrm{B}_1 \frac{\partial V}{\partial y_i} + \mathrm{C}_1 \frac{\partial V}{\partial z_i} \Big) \Big] \delta N_i \\
&\qquad - P \big[\cos(P, x)\, \delta x + \cos(P, y)\, \delta y + \cos(P, z)\, \delta z \big] \Big\} dS_1 \\
&+ \text{etc.} \\
&+ \mathrm{S}_{S_{12}} \Big[2\pi\varepsilon \big(\mathrm{A}_1 \cos(N_1, x) + \mathrm{B}_1 \cos(N_1, y) + \mathrm{C}_1 \cos(N_1, z) \big)^2 \\
&\quad - 2\pi\varepsilon \big(\mathrm{A}_2 \cos(N_2, x) + \mathrm{B}_2 \cos(N_2, y) + \mathrm{C}_2 \cos(N_2, z) \big)^2 \\
&\quad - \varepsilon \Big(\mathrm{A}_1 \frac{\partial V}{\partial x_1} + \mathrm{B}_1 \frac{\partial V}{\partial y_1} + \mathrm{C}_1 \frac{\partial V}{\partial z_1} - \mathrm{A}_2 \frac{\partial V}{\partial x_2} - \mathrm{B}_2 \frac{\partial V}{\partial y_2} - \mathrm{C}_2 \frac{\partial V}{\partial z_2} \Big) \Big] \delta N_1\, dS_{12}. \qquad (9)
\end{aligned}
$$

Dans cette égalité, on a posé, pour abréger,

$$
\left.
\begin{aligned}
\Delta \mathrm{A} &= \delta \mathrm{A} - (\mathrm{B}\omega'' - \mathrm{C}\omega'), \\
\Delta \mathrm{B} &= \delta \mathrm{B} - (\mathrm{C}\omega - \mathrm{A}\omega''), \\
\Delta \mathrm{C} &= \delta \mathrm{C} - (\mathrm{A}\omega' - \mathrm{B}\omega).
\end{aligned}
\right\} \qquad (10)
$$

L'égalité (9) est générale; supposons maintenant vérifiées les égalités (15), (16), (17) et (21) du Chapître précédent, que nous savons être nécessaires pour l'équilibre du système ; il est aisé de voir que ces égalités expriment simplement que l'on a *identiquement*

$$
\begin{aligned}
&\varepsilon\int_1\Big[\frac{\partial V}{\partial x}(\mathrm{B}_1\omega''-\mathrm{C}_1\omega')+\frac{\partial V}{\partial y}(\mathrm{C}_1\omega-\mathrm{A}_1\omega'')+\frac{\partial V}{\partial z}(\mathrm{A}_1\omega'-\mathrm{B}_1\omega)\Big]\,dv_1\\
&+\int_1\Big\{\Big[D_1\frac{\partial\Psi}{\partial x}-\varepsilon\Big(\frac{\partial V}{\partial x}\frac{\partial \mathrm{A}_1}{\partial x}+\frac{\partial V}{\partial y}\frac{\partial \mathrm{B}_1}{\partial x}+\frac{\partial V}{\partial z}\frac{\partial \mathrm{C}_1}{\partial x}\Big)\Big]\,\delta x\\
&\qquad+\Big[D_1\frac{\partial\Psi}{\partial y}-\varepsilon\Big(\frac{\partial V}{\partial y}\frac{\partial \mathrm{A}_1}{\partial y}+\frac{\partial V}{\partial y}\frac{\partial \mathrm{B}_1}{\partial y}+\frac{\partial V}{\partial z}\frac{\partial \mathrm{C}_1}{\partial y}\Big)\Big]\,\delta y\\
&\qquad+\Big[D_1\frac{\partial\Psi}{\partial z}-\varepsilon\Big(\frac{\partial V}{\partial x}\frac{\partial \mathrm{A}_1}{\partial z}+\frac{\partial V}{\partial y}\frac{\partial \mathrm{B}_1}{\partial z}+\frac{\partial V}{\partial z}\frac{\partial \mathrm{C}_1}{\partial z}\Big)\Big]\,\delta z\Big\}\,dv_1\\
&+\mathrm{S}_{s_1}\Big\{\Big[2\pi\varepsilon\,(\mathrm{A}_1\cos(N_i,x)+\mathrm{B}_1\cos(N_i,y)+\mathrm{C}_1\cos(N_i,z))^2\\
&\qquad-\varepsilon\Big(\mathrm{A}_1\frac{\partial V}{\partial x_i}+\mathrm{B}_1\frac{\partial V}{\partial y_i}+\mathrm{C}_1\frac{\partial V}{\partial z_i}\Big)\Big]\,\delta N_i\\
&\qquad-P\,[\cos(P,x)\,\delta x+\cos(P,y)\,\delta y+\cos(P,z)\,\delta z]\Big\}\,dS_1\\
&+\text{etc.}\\
&+\mathrm{S}_{s_{12}}\Big[2\pi\varepsilon\,(\mathrm{A}_1\cos(N_1,x)+\mathrm{B}_1\cos(N_1,y)+\mathrm{C}_1\cos(N_1,z))^2\\
&\qquad-2\pi\varepsilon\,(\mathrm{A}_2\cos(N_2,x)+\mathrm{B}_2\cos(N_2,y)+\mathrm{C}_2\cos(N_2,z))^2\\
&\qquad-\varepsilon\Big(\mathrm{A}_1\frac{\partial V}{\partial x_1}+\mathrm{B}_1\frac{\partial V}{\partial y_1}+\mathrm{C}_1\frac{\partial V}{\partial z_1}-\mathrm{A}_2\frac{\partial V}{\partial x_2}-\mathrm{B}_2\frac{\partial V}{\partial y_2}-\mathrm{C}_2\frac{\partial V}{\partial z_2}\Big)\Big]\,\delta N_1\,dS_{12}\\
&=\int_1\Pi_1\Big(\frac{\partial\delta x}{\partial x}+\frac{\partial\delta y}{\partial y}+\frac{\partial\delta z}{\partial z}\Big)\,dv_1\\
&+\int_2\Pi_2\Big(\frac{\partial\delta x}{\partial x}+\frac{\partial\delta y}{\partial y}+\frac{\partial\delta z}{\partial z}\Big)\,dv_2.
\end{aligned}
$$

Lors donc que les égalités (15), (16), (17) et (21) du Chapître précédent sont supposées vérifiées, l'égalité (9) peut s'écrire :

$$
\begin{aligned}
&\int_1\Big\{\Big[\varepsilon\frac{\partial V}{\partial x}+\frac{\mathrm{A}_1}{f_1(\mathrm{M}_1,D_1)}\Big]\Delta\mathrm{A}_1+\Big[\varepsilon\frac{\partial V}{\partial y}+\frac{\mathrm{B}_1}{f_1(\mathrm{M}_1,D_1)}\Big]\Delta\mathrm{B}_1\\
&\qquad\qquad+\Big[\varepsilon\frac{\partial V}{\partial z}+\frac{\mathrm{C}_1}{f_1(\mathrm{M}_1,D_1)}\Big]\Delta\mathrm{C}_1\Big\}\,dv_1\\
&+\int_1\Big[\phi_1(D_1)-D_1\frac{d\phi_1(D_1)}{dD_1}\\
&\qquad+f_1(\mathrm{M}_1,D_1)-D_1\frac{\partial f_1(\mathrm{M}_1,D_1)}{\partial D_1}+\Pi_1\Big]\Big(\frac{\partial\delta x}{\partial x}+\frac{\partial\delta y}{\partial y}+\frac{\partial\delta z}{\partial z}\Big)\,dv_1\\
&+\text{etc.}=0. \qquad (11)
\end{aligned}
$$

Il est évident que la dilatation cubique $\left(\frac{\partial \delta x}{\partial x} + \frac{\partial \delta y}{\partial y} + \frac{\partial \delta z}{\partial z}\right)$ a une valeur arbitraire en tout point des fluides 1 et 2. δA_1, δB_1, δC_1, ayant des valeurs arbitraires, il en est de même, en vertu des égalités (10) de ΔA_1, ΔB_1, ΔC_1. Donc, pour que l'égalité (11) ait lieu, il faut et il suffit :

1°. Que l'on ait, *en tout point du fluide* 1,

$$\left.\begin{aligned} A_1 &= -\varepsilon f_1(M_1, D_1)\frac{\partial V}{\partial x}, \\ B_1 &= -\varepsilon f_1(M_1, D_1)\frac{\partial V}{\partial y}, \\ C_1 &= -\varepsilon f_1(M_1, D_1)\frac{\partial V}{\partial z} \end{aligned}\right\} \qquad (12)$$

et, *en tout point du fluide* 2,

$$\left.\begin{aligned} A_2 &= -\varepsilon f_2(M_2, D_2)\frac{\partial V}{\partial x}, \\ B_2 &= -\varepsilon f_2(M_2, D_2)\frac{\partial V}{\partial y}, \\ C_2 &= -\varepsilon f_2(M_2, D_2)\frac{\partial V}{\partial z}. \end{aligned}\right\} \qquad (12\text{ bis})$$

2°. Que l'on ait, *en tout point du fluide* 1,

$$\phi_1(D_1) - D_1\frac{d\phi_1(D_1)}{dD_1} + f_1(M_1, D_1) - D_1\frac{\partial f_1(M_1, D_1)}{\partial D_1} + \Pi_1 = 0, \qquad (13)$$

et, *en tout point du fluide* 2,

$$\phi_2(D_2) - D_2\frac{d\phi_2(D_2)}{dD_2} + f_2(M_2, D_2) - D_2\frac{\partial f_2(M_2, D_2)}{\partial D_2} + \Pi_2 = 0. \qquad (13\text{ bis})$$

Si l'on observe que les conditions (12) et (12 bis) entraînent les égalités (15) du Chapître précédent, on voit que *les conditions nécessaires et suffisantes pour l'équilibre d'un système de fluides compressibles et dénués de force coercitive, sont données par les égalités* (12), (12 bis), (13), (13 bis) *du présent Chapître, jointes aux égalités* (16), (17) *et* (21) *du Chapître précédent.*

Les égalités (12) et (12 bis) ont joué un rôle fondamental dans toutes nos études sur les corps magnétiques ou diélectriques; quant aux égalités (13) et (13 bis), nous avons signalé leur importance dans nos *Leçons sur l'Electricité et le Magnétisme.*

On Ternary Substitution-Groups of Finite Order which leave a Triangle unchanged.

BY H. MASCHKE.

In his papers, "Sur les équations différentielles linéaires à intégrale algébrique,"* and "Sur la détermination des groupes d'ordre fini contenues dans le groupe linéaire,"† C. Jordan has enumerated all those ternary linear substitution-groups whose order is a finite number. Three of these groups, being of special interest, viz. one group of order 60, isomorphic with the icosahedron-group,‡ one of order 216, the so-called Hessian-group,§ and one of order 168,|| have been thoroughly investigated. But nothing has been done as yet with regard to those apparently simple ternary groups whose substitutions are given by formulæ of this kind:

$$z_1' = az_i, \quad z_2' = bz_k, \quad z_3' = cz_l,$$

where a, b, c are roots of unity and i, k, l in some order equal to $1, 2, 3$.

It seems to be appropriate to name substitutions of this type "*monomial*" substitutions, and groups containing only monomial substitutions "*monomial groups.*"

In the following a first step towards a complete treatment of these ternary monomial groups will be made, viz. the investigation of those groups "G" whose substitutions are generated by the following two monomial substitutions:

$$S\colon \left.\begin{aligned} z_1' &= z_2, \\ z_2' &= z_3, \\ z_3' &= z_1, \end{aligned}\right\} (1), \qquad T\colon \left.\begin{aligned} z_1' &= a_1 z_1, \\ z_2' &= a_2 z_2, \\ z_3' &= a_3 z_3, \end{aligned}\right\} \qquad (2)$$

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* Borchhardt's Journal, Bd. 84.

† Atti della Reale Accademia di Napoli, 1880.

‡ F. Klein, Math. Ann., Bd. 12, p. 529; Icosaëder, p. 213 ff.

§ A. Witting, Dissertation, Göttingen, 1887, p. 23 ff.; H. Maschke, Math. Ann., Bd. 33, p. 324.

|| F. Klein, Math. Ann., Bd. 14, p. 144; Bd. 15, p. 265. Klein-Fricke, Modulfunctionen, I, p. 692 ff.

where a_1, a_2, a_3 are roots of unity. T may also be written in this form:

$$\left.\begin{aligned} z_1' &= e^{\frac{2k_1\pi i}{m_1}} z_1, \\ z_2' &= e^{\frac{2k_2\pi i}{m_2}} z_2, \\ z_3' &= e^{\frac{2k_3\pi i}{m_3}} z_3, \end{aligned}\right\} \tag{3}$$

k_1, k_2, k_3 and m_1, m_2, m_3 being integers.

The determinant of substitution S is 1. We impose the same condition also on substitution T, viz. $a_1a_2a_3 = 1$, or

$$\frac{k_1}{m_1} + \frac{k_2}{m_2} + \frac{k_3}{m_3} = n, \tag{4}$$

where n is any integer.

§1. *The Invariant Forms of G.*

We may assume k_1 to be prime to m_1, k_2 to m_2, k_3 to m_3. Let R be the greatest common divisor of m_1 and m_2, so that

$$m_1 = p_1R, \quad m_2 = p_2R, \tag{5}$$

then p_1 and p_2 will be prime to each other and

$$p_1p_2R = P \tag{6}$$

the least common multiple of m_1 and m_2. But equation (4) shows that P is divisible also by m_3, therefore *P is the period of T.*

It follows from S that in every *invariant form* of G—i. e. a homogeneous integral function of z_1, z_2, z_3 which remains absolutely unchanged when the three variables are operated upon by the substitutions of G—only terms of these three types are admissible:

Type I : $z_1^\alpha + z_2^\alpha + z_3^\alpha$,
Type II : $z_1^\alpha z_2^\beta + z_2^\alpha z_3^\beta + z_3^\alpha z_1^\beta$,
Type III: $z_1^\alpha z_2^\beta z_3^\gamma + z_2^\alpha z_3^\beta z_1^\gamma + z_3^\alpha z_1^\beta z_2^\gamma$.

We see at once that $z_1z_2z_3$ is invariant, or, spoken geometrically, that the triangle of reference remains unchanged. Hence every form of type III is reducible to a product of some power of $z_1z_2z_3$ into a form of type I or type II, and if in type II we admit the value zero for α or β, we see that forms of type I may also be discarded. So we have the following theorem:

Every invariant form of G is an integral function of $z_1z_2z_3$ and of forms $z_1^\alpha z_2^\beta + z_2^\alpha z_3^\beta + z_3^\alpha z_1^\beta$ (α and β being positive integers or zero).

We now have to find the conditions that $z_1^\alpha z_2^\beta + z_2^\alpha z_3^\beta + z_3^\alpha z_1^\beta$ remains invariant with regard to T.

If we apply T to $z_1^\alpha z_2^\beta + z_2^\alpha z_3^\beta + z_3^\alpha z_1^\beta$ and put this expression equal to the transformed expression, we see that the following three equations must be satisfied:

$$\left.\begin{aligned} \frac{\alpha k_1}{m_1} + \frac{\beta k_2}{m_2} &= \lambda\,, \\ \frac{\alpha k_2}{m_2} + \frac{\beta k_3}{m_3} &= \lambda_1, \\ \frac{\alpha k_3}{m_3} + \frac{\beta k_1}{m_1} &= \lambda_2 \end{aligned}\right\} \tag{7}$$

(λ, λ_1, λ_2 integers).

Substituting, now, the value of $\frac{k_3}{m_3}$ taken from (4) into (7), we obtain

$$\left.\begin{aligned} \frac{\alpha k_1}{m_1} + \frac{\beta k_2}{m_2} &= \lambda, \\ \frac{\alpha k_2}{m_2} - \beta\left(\frac{k_1}{m_1} + \frac{k_2}{m_2}\right) &= \mu \end{aligned}\right\} \tag{8}$$

(λ and μ integers).

Adding together these two equations (8) and combining the result with (4), we obtain the third equation (7), which therefore may be omitted.

Let now c be the greatest common divisor of k_1 and k_2, so that

$$k_1 = c\kappa_1,\quad k_2 = c\kappa_2, \tag{9}$$

then λ and μ must be divisible by c because k_1 is prime to m_1 and k_2 to m_2. Changing the signification of λ and μ, we have these two equations

$$\begin{aligned} \frac{\alpha \kappa_1}{m_1} + \frac{\beta \kappa_2}{m_2} &= \lambda, \\ \frac{\alpha \kappa_2}{m_2} - \beta\left(\frac{\kappa_1}{m_1} + \frac{\kappa_2}{m_2}\right) &= \mu \end{aligned}$$

(λ and μ integers).

Multiplying by $m_1 . m_2$ and substituting the expressions (5), we obtain

$$\left.\begin{aligned} \alpha\varkappa_1 p_2 + \beta\varkappa_2 p_1 \qquad\quad &= \lambda p_1 p_2 R, \\ \alpha\varkappa_2 p_1 - \beta(\varkappa_1 p_2 + \varkappa_2 p_1) &= \mu p_1 p_2 R, \end{aligned}\right\} \tag{10}$$

and in these equations $\varkappa_1$ is prime to $\varkappa_2$ and to p_1, $\varkappa_2$ to $\varkappa_1$ and p_2, p_1 to p_2. It follows, therefore, from the first of these equations that α is a multiple of p_1, β a multiple of p_2, and then from the second that α is a multiple of p_2, and β a multiple of p_1. Hence

$$\alpha = \alpha' p_1 p_2, \quad \beta = \beta' p_1 p_2, \tag{11}$$

where α', β' are two new integers. Substituting these values into (10), we may divide by $p_1 p_2$ and have

$$\begin{aligned} \alpha'\varkappa_1 p_2 + \beta'\varkappa_2 p_1 &= \lambda R, \\ \alpha'\varkappa_2 p_1 - \beta'(\varkappa_1 p_2 + \varkappa_2 p_1) &= \mu R, \end{aligned}$$

or, using the abbreviation

$$\varkappa_1 p_2 = p, \quad \varkappa_2 p_1 = q, \tag{12}$$

$$\left.\begin{aligned} \alpha' p + \beta' q \qquad\quad &\equiv 0 \pmod{R}, \\ \alpha' q - \beta'(p+q) &\equiv 0 \pmod{R}, \end{aligned}\right\} \tag{13}$$

where p and q are prime to each other.

Solving (13) with regard to α' and β', we find

$$\left.\begin{aligned} \alpha'(p^2 + pq + q^2) &\equiv 0 \pmod{R}, \\ \beta'(p^2 + pq + q^2) &\equiv 0 \pmod{R}. \end{aligned}\right\} \tag{14}$$

Let now t be the greatest common divisor of $p^2 + pq + q^2$ and R, so that

$$\left.\begin{aligned} p^2 + pq + q^2 &= st, \\ R &= rt, \end{aligned}\right\} \tag{15}$$

then it follows from (14),

$$\begin{aligned} \alpha' s &\equiv 0 \pmod{r}, \\ \beta' s &\equiv 0 \pmod{r}. \end{aligned}$$

But r and s have no common divisor, therefore

$$\alpha' = r\alpha'', \quad \beta' = r\beta'', \tag{16}$$

where α'' and β'' are again two new integers.

Substituting (16) into (13), we may divide by r and obtain

$$\alpha'' p + \beta'' q \qquad\quad \equiv 0 \pmod{t}, \tag{17}$$

$$\alpha'' q - \beta''(p+q) \equiv 0 \pmod{t}, \tag{18}$$

and here t *has no common divisor with* p *and* q. This follows immediately from $p^2 + pq + q^2 = st$ and from p being prime to q.

We now have to find the solutions of the congruences (17) and (18). For that purpose let us first treat congruence (17). Assign to α'' any positive integral value. We may write

$$\alpha'' \equiv n \pmod{t}, \tag{19}$$

where $n < t$. Then we have

$$\beta'' q \equiv -np \pmod{t}. \tag{20}$$

Solve now the congruence

$$v . q \equiv -p \pmod{t}. \tag{21}$$

We know there will exist one and only one solution $v < t$. Denote this particular value by v. Then the general solution of (20) is

$$\beta'' \equiv nv \pmod{t}. \tag{22}$$

We have thus in (19) and (22) the most general solution of (17).

Let us substitute these values of α'' and β'' into (18). We obtain

$$nq - nv(p+q) \equiv 0 \pmod{t}.$$

If $n = 0$, this congruence is satisfied. If $n \neq 0$, we may divide by n and multiply by q (which is prime to t),

$$\therefore q^2 - vq(p+q) \equiv 0 \pmod{t}.$$

But $vq \equiv -p \pmod{t}$, hence (18) becomes

$$q^2 + pq + p^2 \equiv 0 \pmod{t},$$

which congruence is satisfied indeed according to (15). *The solutions* α'' *and* β'' *of* (17) *satisfy therefore also* (18) *without any further restriction.*

It is evident that the general solution may also be obtained by first assuming some integral value for β'', i. e.

$$\beta'' \equiv n \pmod{t}. \tag{23}$$

Denoting the smallest positive root of the congruence

$$w . p \equiv -q \pmod{t} \tag{24}$$

by w, we have

$$\alpha'' \equiv nw \pmod{t}. \tag{25}$$

The values of α and β are now given by (11), (16) and (19), (20) or (25), (23), viz.

$$\left.\begin{aligned}\alpha &= p_1p_2r(\lambda t + n),\\ \beta &= p_1p_2r(\mu t + nv),\end{aligned}\right\} \tag{26}$$

or

$$\left.\begin{aligned}\alpha &= p_1p_2r(\lambda t + nw),\\ \beta &= p_1p_2r(\mu t + n).\end{aligned}\right\} \tag{27}$$

Since $p_1p_2rt = p_1p_2R = P$—see equations (15) and (6)—we may also write

$$\left.\begin{aligned}\alpha &\equiv n.p_1p_2r \pmod{P},\\ \beta &\equiv nv.p_1p_2r \pmod{P},\end{aligned}\right\} \tag{28}$$

or

$$\left.\begin{aligned}\alpha &\equiv nw.p_1p_2r \pmod{P},\\ \beta &\equiv n.p_1p_2r \pmod{P},\end{aligned}\right\} \tag{29}$$

where n stands for any positive integer $< t$, or zero.

We have thus obtained the following final result:

All invariant forms of G are integral functions of $z_1z_2z_3$ and of forms $z_1^\alpha z_2^\beta + z_2^\alpha z_3^\beta + z_3^\alpha z_1^\beta$, where α and β are to be found by the following process: Find the greatest common divisor R of m_1 and m_2, and put $m_1 = p_1R$, $m_2 = p_2R$, $p_1p_2R = P$. Divide the two numbers k_1, k_2 by their greatest common divisor and call the quotients x_1 and x_2. Denote $p_1x_2 = p$ and $p_2x_1 = q$. Let t be the greatest common divisor of R and the quadratic form $p^2 + pq + q^2$; put $R = rt$. Solve the congruences

$$vq \equiv -p \pmod{t} \text{ and } wp \equiv -q \pmod{t},$$

then α and β are given by formulæ (28) *or* (29).

§2. *The Quantities v, w and t.*

In the following we shall always suppose $t > 1$. The case $t = 1$ will be treated separately under the head "special cases" in §4.

Let us multiply the congruence (21) by (24). We may then divide by pq, which is prime to t, and obtain

$$vw \equiv 1 \pmod{t}. \tag{30}$$

Multiplying now (21) and (24) by p and q respectively and adding together, we obtain

$$(v + w)pq \equiv -p^2 - q^2 \pmod{t},$$

or, since $-p^2-q^2\equiv pq \pmod{t}$, see (15),

$$v+w\equiv 1 \pmod{t}.$$

But v and w are both supposed to be $<t$, hence we have

$$v+w=t+1. \tag{31}$$

Finally we deduce from (21) the two congruences

$$v^2q^2\equiv p^2 \pmod{t},$$
$$-vq^2\equiv pq \pmod{t},$$

and joining the identity $q^2\equiv q^2 \pmod{t}$, we obtain by addition

$$q^2(v^2-v+1)\equiv p^2+pq+q^2 \pmod{t},$$

or

$$v^2-v+1\equiv 0 \pmod{t}. \tag{32}$$

In a similar manner we find

$$w^2-w+1\equiv 0 \pmod{t}. \tag{33}$$

Thus we have the following result: The two quantities v and w are roots of the quadratic congruence

$$\omega^2-\omega+1\equiv 0 \pmod{t}; \tag{34}$$

they are connected by the relations

$$v+w=t+1 \text{ and } vw\equiv 1 \pmod{t}.$$

The congruence (34) is not solvable for every integral value of t. But it is always solvable for those values of t which occur in the present problem. It is shown in the Theory of Numbers[*] that the quadratic form p^2+pq+q^2 (p and q prime to each other) represents all those and only those numbers N which are given by $p_1^{\lambda_1}.p_2^{\lambda_2}.p_3^{\lambda_3}\ldots$. or $3p_1^{\lambda_1}.p_2^{\lambda_2}.p_3^{\lambda_3}\ldots$. where $p_1, p_2, p_3\ldots$. are prime numbers of the form $3h+1$. The number t, being a divisor of p^2+pq+q^2 is therefore also of the form N. The smallest possible values of t are these: 1, 3, 7, 13, 19, 21, 31, 37, 39, 43, 49, 57, 61, 67, 73, 79, 91, 93, 97, etc.

Let v be some root of the congruence (34), then $w=t+1-v$ is also a root of the same congruence, and the relation $vw\equiv 1 \pmod{t}$ is satisfied too. If t is of the form p_1^{λ} or $3p_1^{\lambda}$, then the congruence (34) has only two roots, and these roots are to be taken as v and w. The lowest value of t for which (34) has more than two roots is $t = 91$. In this case we have the four roots 10, 17, 75, 82, and here either 10 and 82 or 17 and 75 may be taken as a system v, w.

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[*] Dirichlet-Dedekind, Zahlentheorie, 1871, p. 165.

Let us examine the case $v = w$. In this case we have from (31)

$$v = w = \tfrac{1}{2}(t + 1),$$

and from (32),

$$(t + 1)^2 - 2(t + 1) + 4 \equiv 0 \pmod{t},$$

or

$$t^2 + 3 \equiv 0 \pmod{t},$$

hence $t = 3$, $v = w = 2$.

The two numbers v and w are therefore always distinct except in the case $t = 3$.

§3. *Connection between the Invariant Forms.*

For further investigation of the invariant forms let us put

$$p_1 p_2 r = \vartheta, \tag{35}$$

$$z^{p_1 p_2 r} = z^{\vartheta} = y, \tag{36}$$

and let us denote for shortness

$$\begin{aligned} y_1^a y_2^b + y_2^a y_3^b + y_3^a y_1^b &= (y_1^a y_2^b), \\ y_1^a y_2^b y_3^c + y_2^a y_3^b y_1^c + y_3^a y_1^b y_2^c &= (y_1^a y_2^b y_3^c). \end{aligned}$$

All invariant forms of G are then given as integral functions of $z_1 z_2 z_3$ and expressions of the form

$$(y_1^{\lambda t + n} \cdot y_2^{\mu t + nv}). \tag{37}$$

Furthermore, we shall use the following notations:

$$\left.\begin{aligned} y_1 y_2 y_3 &= A, \\ y_1^t + y_2^t + y_3^t &= E. \end{aligned}\right\} \tag{38}$$

If in (37) n runs from 1 to $t - 1$, then nv will constitute a complete system of residues (mod. t). Let us denote

$$nv \equiv v_n \pmod{t}, \tag{39}$$

where $v_n < t$. Then v_n will assume all the values from 1 to $t - 1$ in some order when n runs from 1 to $t - 1$. Thus we obtain $t - 1$ forms,

$$(y_1^n y_2^{v_n}), \qquad (n = 1, 2, 3, \ldots . t - 1),$$

which are contained in (37). We shall denote these special forms by

$$\psi_n = (y_1^n y_2^{v_n}), \tag{40}$$

or simply by ψ when no reference to the value of n is needed.

The same system of forms (40) is also, according to (27), given by

$$\psi_{w_n} = (y_1^{w_n} y_2^n), \qquad (n = 1, 2, 3, \ldots . t - 1),$$

where again $w_n < t$ is defined by the congruence

$$nw \equiv w_n \pmod{t}. \tag{41}$$

We now propose to show *that all the invariant forms given by* (37) *are expressible in terms of* A, E *and* ψ_n.

We know that v will be distinct from w unless $t = 3$. To fix the ideas, let us in the following always assume $v > w$. If in a given case w, defined by (24), should be greater than v, we have only to interchange v and w or α and β. We prove now the following lemmas:

1). $y_1^t y_2^t + y_2^t y_3^t + y_3^t y_1^t = (y_1^t y_2^t)$ *is expressible in terms of* A, E *and* ψ.

From $\quad v^2 \equiv v - 1 \pmod{t}$

it follows that $\quad v_v = v - 1,$

and likewise $\quad w_w = w - 1.$

Hence $\quad \psi_v = (y_1^v y_2^{v-1})$ and $\psi_w = (y_1^w y_2^{w-1}).$ $\quad$ (42)

Multiplying now $\psi_v . \psi_{w-1}$ we find

$$\psi_v \psi_{w-1} = (y_1^{v+w-1} y_2^{v+w-1}) + A^w (y_1^{v-w} y_2^{v-2}) + A^{w-1} (y_1^{v+1} y_2^{v-w}).$$

But $v + w - 1 = t$, see (31), hence

$$(y_1^t y_2^t) = \psi_v \psi_{w-1} - A^w \psi_{v-w} - A^{w-1} \psi_{v+1}. \tag{43}$$

This deduction fails for $t = 3$. In this case we find directly

$$(y_1^3 y_2^3) = \psi_1 \psi_2 - 3A^3 - AE. \tag{44}$$

2). $(y_1^{t+n} y_2^{v_n})$ *is expressible in terms of* $(y_1^n y_2^{t+v_n})$, A, E *and* ψ.

This follows immediately by performing the multiplication

$$(y_1^n y_2^{v_n})(y_1^t + y_2^t + y_3^t) = (y_1^{t+n} y_2^{v_n}) + (y_1^n y_2^{t+v_n}) + (y_1^n y_2^{v_n} y_3^t),$$

whence $\quad (y_1^{t+n} y_2^{v_n}) = -(y_1^n y_2^{t+v_n}) + E . \psi_n - A^n \psi_{v_n - n},$ $\quad$ (45)

if $n < v_n$. If, however, $n > v_n$, the last term in (45) is to be replaced by $A^{v_n} \psi_{n - v_n}$.

3). $(y_1^{t+n}y_2^{t+v_n})$ *is expressible in terms of* $(y_1^n y_2^{t+v_n})$, A, E *and* ψ.

We find

$$\begin{aligned}(y_1^n y_2^{v_n})(y_1^t y_2^t) &= (y_1^{t+n}y_2^{t+v_n}) + (y_1^n y_2^{t+v_n} y_2^t) + (y_1^{t+n} y_2^{v_n} y_2^t)\\ &= (y_1^{t+n}y_2^{t+v_n}) + A^n(y_1^{t+v_n-n}y_2^{t-n}) + A^{v_n}(y_1^{t-v_n}y_2^{t+n-v_n}),\end{aligned}$$

or

$$(y_1^{t+n}y_2^{t+v_n}) = E.\psi_n - A^n(y_1^{t+v_n-n}y_2^{t-n}) + A^{v_n}(y_1^{t-v_n}y_2^{t+n-v_n}). \tag{46}$$

If $v_n > n$, then the factor of A^{v_n} in (46) is a function ψ_n, and the factor of A^n, being of the type $(y_1^{t+m}y_2^{v_m})$, is reducible to $(y_1^m y_2^{t+v_m})$ and A, E, ψ_n, according to (45). If, however, $n > v_n$, then the factor of A^n is a function ψ_n and the factor of A^{v_n} a function $(y_1^n y_2^{t+v_n})$.

4). $(y_1 y_2^{t+v})$ *is expressible in terms of* A, E *and* ψ.

From

$$(y_1^w y_2)(y_1^v y_2^{v-1}) = (y_1^{t+1}y_2^v) + A(y_1^{v-1}y_2^{t-1}) + A^w(y_1^{v+1-w}y_2^{v-w+1})$$

we obtain

$$(y_1^{t+1}y_2^v) = \psi_w\psi_v - A\psi_{v-1} - A^w p_{v+1-w},$$

and from (45), putting $n = 1$,

$$(y_1^{t+1}y_2^v) = -(y_1 y_2^{t+v}) + E\psi_1 - A\psi_{v-1}.$$

Combining the last two equations, we have

$$(y_1 y_2^{t+v}) = E\psi_1 - \psi_v\psi_w + A^w\psi_{v+1-w}. \tag{47}$$

This deduction fails again for $t = 3$. In this case we find

$$(y_1 y_2^5) = (A + E)\psi_1 - \psi_2^2. \tag{48}$$

5). $(y_1^n y_2^{t+v_n})$ *is expressible in terms of* A, E *and* ψ.

This theorem has already been proved for $n = 1$ (see lemma 4). In order to prove it for functions $(y_1^{n+1}y_2^{t+v_{n+1}})$ $(n = 1, 2, 3, \ldots. t - 2)$, we have to distinguish two cases, viz. $v + v_n = t + v_{n+1}$ and $v + v_n = v_{n+1}$. In the first case let us form the product

$$(y^n y_2^{v_n})(y_1 y_2^v) = (y_1^{n+1}y_2^{t+v_{n+1}}) + A\Psi,$$

whence

$$(y_1^{n+1}y_2^{t+v_{n+1}}) = \psi_1\psi - A\Psi, \tag{49}$$

in the latter,

$$(y_1^n y_2^{v_n})(y_1 y_2^{t+v}) = (y_1^{n+1}y_2^{t+v_{n+1}}) + A\Psi',$$

whence

$$(y_1^{n+1}y_2^{t+v_{n+1}}) = (y_1 y_2^{t+v})\psi_a - A\Psi',$$

or, applying (47),

$$(y_1^{n+1}y_2^{t+v_{n+1}}) = E\psi_1\psi_n - \psi_n\psi_v\psi_w - A\Psi'', \tag{50}$$

where Ψ, Ψ' and Ψ'' are expressions consisting of functions of the type $(y_1^{t+m}y_2^{t+v_m})$, $(y_1^{t+p}y_2^{v_p})$, $(y_1^n y_2^{t+v_n})$, E, ψ and of products of some powers of A into such functions. But $(y_1^{t+m}y_2^{t+v_m})$ and $(y_1^{t+p}y_2^{v_p})$ can be reduced to A, E, ψ, $(y_1^n y_2^{t+v_n})$ according to lemma 2 and 3, and let us suppose that this reduction has actually been made. Ψ, Ψ' *and* Ψ'' *contain then only functions* A, E, ψ *and* $(y_1^n y_2^{t+v_n})$. The degrees of all the $t-2$ functions $(y_1^{n+1}y_2^{t+v_{n+1}})$ will be distinct from each other, for any equation

$$n+1+t+v_{n+1}=m+1+t+v_{m+1}$$

would imply $\quad (n+1)(v+1)\equiv(m+1)(v+1) \pmod{t},$

or $n\equiv m \pmod{t}$, i. e. $n=m$. (The case $v+1\equiv 0 \pmod{t}$ leads to $v=t-1$, $w=2$, $t=3$, which shall be excluded.) Let, then, $(y_1^{m+1}y_2^{t+v_{m+1}})$ be that function whose degree is a minimum and now apply formula (49) or (50). The degree of Ψ or of Ψ'' is $m+1+t+v_{m+1}-3$. It follows that Ψ and Ψ'' cannot contain functions of the type $(y_1^n y_2^{t+v_n})$ at all since $m+1+t+v_{m+1}-3$ is smaller than the smallest possible degree of $(y_1^n y_2^{t+v_n})$. Therefore $(y_1^{m+1}y_2^{t+v_{m+1}})$ must be expressible in terms of A, E and ψ only. We now apply the same conclusion to the successive functions $(y_1^{n+1}y_2^{t+v_{n+1}})$ which we arrange according to their dimensions. The functions Ψ and Ψ'' occurring in that representation will then contain either no functions $(y_1^n y_2^{t+v_n})$ at all or only those which have already been reduced to A, E and ψ. Hence we have the theorem: *Every function* $(y_1^n y_2^{t+v_n})$ *is expressible in terms of* A, E *and* ψ.

The case $t=3$ has again to be treated separately. We find directly

$$(y_1^2 y_2^4)=\psi_1^2-2A\psi_2. \tag{51}$$

Combining now lemmas 1, 2, 3, 4, 5, we have the following result: *The functions* $(y_1^n y_2^{t+v_n})$, $(y_1^{t+n}y_2^{v_n})$, $(y_1^{t+n}y_2^{t+v_n})$ *for all the values of* $n=0, 1, 2, \ldots . t-1$ *are expressible in terms of* A, E *and* ψ.

Multiplying these functions by E and $y_1^t y_2^t + y_2^t y_3^t + y_3^t y_1^t = B$, we can increase either exponent by t. We find

$$(y_1^n y_2^{2t+v_n}) = (y_1^n y_2^{t+v_n})E - (y_1^{t+n}y_2^{t+v_n}) - A^n(y_1^{t+v_n-n}y_2^{t-n}), \tag{52}$$

and a corresponding formula for $(y_1^{2t+n}y_2^{v_n})$,

$$(y_1^{t+n}y_2^{2t+v_n}) = (y_1^n y_2^{t+v_n})B - A^n(y_1^{2t+v_n-n}y_2^{t-n}) - A^t\psi_n, \tag{53}$$

and a corresponding formula for $(y_1^{2t+n}y_2^{t+v_n})$,

$$(y_1^{2t+n}y_2^{2t+v_n}) = (y_1^{t+n}y_2^{t+v_n})B - A^t(y_1^n y_2^{t+v_n}) - A^t(y_1^{t+n}y_2^{v_n}). \tag{54}$$

This method can be continued until all the functions $(y_1^{\lambda t+n} y_2^{\mu t+v_n})$, where $\lambda, \mu = 0, 1, 2, 3, \ldots.$ and $n = 1, 2, 3, \ldots. t-1$, are expressed in terms of A, B, E and ψ, i. e. in terms of A, E and ψ, since B itself is given in terms of A, E and ψ, see (43) and (44).

The case $n = 0$ requires a special deduction. If $\lambda = \mu$, then the function $(y_1^{\lambda t} y_2^{\lambda t})$ is symmetrical, and therefore expressible in terms of

$$\begin{aligned} y_1^t + y_2^t + y_3^t &= E, \\ y_1^t y_2^t + y_2^t y_3^t + y_3^t y_1^t &= B, \\ y_1^t y_2^t y_3^t &= A^t. \end{aligned}$$

If $\lambda \neq \mu$ it becomes necessary to express the product of differences,

$$(y_1^t - y_2^t)(y_2^t - y_3^t)(y_3^t - y_1^t) = (y_1^t y_2^{2t}) - (y_1^{2t} y_2^t).$$

We obtain this by performing the multiplications

$$(y_1^v y_2^{t+v-1}).(y_1^{w-1} y_2^w) \text{ and } (y_1^{t+w-1} y_2^w)(y_1^v y_2^{v-1}).$$

The result is:

$$(y_1^t y_2^{2t}) = \psi_{w-1}(y_1^v y_2^{t+v-1}) - A^w (y_1^{v-w} y_2^{t+v-2}) - A^{w-1}(y_1^{v+1} y_2^{t+v-w}), \tag{55}$$
$$(y_1^{2t} y_2^t) = \psi_v (y_1^{t+w-1} y_2^w) \quad - A^w (y_1^{v-w} y_2^{t+v-2}) - A^{v-1}(y_1^{t+w-v} y_2^{w+1}). \tag{56}$$

Now, every function $(y_1^{\lambda t} y_2^{\mu t})$ is expressible in terms of the symmetric functions A, B, E and of (55) and (56).

Thus we have the final result:

Denote $r_1 r_2 p = \vartheta$, $y = z^\vartheta$, $y_1 y_2 y_3 = A$, $y_1^t + y_2^t + y_3^t = E$, $(y_1^n y_2^{v_n}) = \psi_n$, *and suppose* $t > 1$, *then every invariant form of* G *is an integral function of* $z_1 z_2 z_3 = \sqrt[\vartheta]{A}$, E and ψ_n $(n = 1, 2, \ldots. t-1)$.

This system of the $t+1$ forms $\sqrt[\vartheta]{A}$, E, ψ_n may be called the "*reduced system*" of invariant forms. It is in general greater than the "*complete system*,"[*] and it contains the forms of the complete system. The results of the next section will show that for $t > 3$ some of the forms ψ_n can be expressed as integral functions of the other forms of the reduced system.

---

[*] A complete system of invariant forms is given by the *minimum* number of invariants such that every other invariant may be expressed as an integral function of the forms of the system.

§4. *Special Cases.*

I. $t=1$.

The case $t=1$ covers a great many special groups G; for instance, all those where R, the greatest common divisor of m_1 and m_2, is either 1, or where it consists of prime factors of the form $3h+2$, i. e. 2, 5, 11, etc.

For $t=1$ the formulæ (26) are reduced to

$$\alpha=\lambda\vartheta \text{ and } \beta=\mu\vartheta,$$

where λ and μ are two integers. All invariant forms are therefore given by $z_1z_2z_3$ and $(z_1^{\lambda\vartheta}z_2^{\mu\vartheta})$; they are expressible as integral functions of

$$\begin{aligned} z_1z_2z_3 &= \sqrt[\vartheta]{A}, \\ y_1+y_2+y_3 &= E, \\ y_1y_2+y_2y_3+y_3y_1 &= B, \\ \text{and} \qquad (y_1-y_2)(y_2-y_3)(y_3-y_1) &= \Delta. \end{aligned}$$

These four forms constitute therefore the complete system of invariants of the group. There exists one relation between them, viz.

$$\Delta^2=18ABE-4AE^3-4B^3+B^2E^2-27A^2.$$

II. $t=3$.

In this case the reduced system consists of

$$\sqrt[\vartheta]{A},\ E,\ \psi_1=(y_1y_2^2),\ \ \psi_2=(y_1^2y_2),$$

which is also the complete system. The relation between its four forms is this:

$$\psi_1^3+\psi_2^3-\psi_1\psi_2(6A+E)+A(9A^2+3AE+E^2)=0.$$

III. $t=7$.

The roots of the congruence

$$\omega^2-\omega+1\equiv 0 \pmod 7$$

are $v=5$, $w=3$. The reduced system consists of

$$\sqrt[\vartheta]{A},\ E \text{ and 6 functions } \psi_n=(y_1^n y_2^{v_n}) \qquad (n=1,2,\ldots .6).$$

Corresponding values of n and v_n are given in the following table:

$n=$	1	2	3	4	5	6
$v_n=$	5	3	1	6	4	2

The complete system consists of $\sqrt[3]{A}$, E, and ψ_1, ψ_2, ψ_3. We find

$$\begin{aligned} \psi_4 &= \psi_2^2 - 2A^2\psi_3, \\ \psi_5 &= \psi_2\psi_3 - A\psi_1 - 3A^3, \\ \psi_6 &= \psi_3^2 - 2A\psi_2. \end{aligned}$$

There must exist two relations between the five forms of the complete system. These are

$$\left.\begin{aligned} &\psi_2^2 - \psi_1\psi_3 + AE - A^2\psi_3 = 0, \\ &\psi_1^2 + \psi_3^3 - E\psi_2 - 5A\psi_2\psi_3 + 3A^2\psi_1 + 9A^4 = 0. \end{aligned}\right\} \tag{57}$$

As an example may serve the group* generated by

$$S: \begin{aligned} z_1' &= z_2, \\ z_2' &= z_3, \\ z_3' &= z_1, \end{aligned} \quad \text{and} \quad T: \begin{aligned} z_1' &= \gamma z_1, \\ z_2' &= \gamma^4 z_2, \\ z_3' &= \gamma^2 z_3, \end{aligned} \text{ where } \gamma = e^{\frac{2\pi i}{7}}.$$

We have here $m_1 = m_2 = 7$; hence $R = 7$, $p_1 = p_2 = 1$, $k_1 = 1$, $k_2 = 4$ and also $\varkappa_1 = 1$, $\varkappa_2 = 4$; $p = \varkappa_1 p_2 = 1$, $q = \varkappa_2 p_1 = 4$, $p^2 + pq + q^2 = 21$, therefore $t = 7$, $r = 1$, $\vartheta = p_1 p_2 r = 1$. The congruence $4v \equiv -1 \pmod 7$ gives $v = 5$. The complete system is therefore given by

$$\left.\begin{aligned} A &= z_1 z_2 z_3 \qquad\qquad\quad , \\ \psi_3 &= z_1^3 z_2 + z_2^3 z_3 + z_3^3 z_1, \\ \psi_2 &= z_1^2 z_2^3 + z_2^2 z_3^3 + z_3^2 z_1^3, \\ \psi_1 &= z_1 z_2^5 + z_2 z_3^5 + z_3 z_1^5, \\ E &= z_1^7 \quad + z_2^7 \quad + z_3^7 \quad . \end{aligned}\right\} \tag{58}$$

IV. $t = 13$.

The roots of the congruence

$$\omega^2 - \omega + 1 \equiv 0 \pmod{13}$$

are $v = 10$, $w = 4$. The reduced system consists of

$$\sqrt[3]{A},\ E, \text{ and 12 functions } \psi_n = (y_1^n y_2^{vn}) \qquad (n = 1, 2, \ldots . 12).$$

---

[*] This group (of order 21) and its invariant forms play an important part in many investigations. The group occurs as a subgroup of the G_{168} mentioned in the introduction, and also as a subgroup of a quaternary G_{168} (see my paper on this group read at the International Mathematical Congress, Chicago, 1893), which latter is contained again as a subgroup in a quaternary group of order $\frac{7!}{2}$ (F. Klein, Ueber Gleichungen 6. und 7. Grades, Math. Ann., Bd. 28, p. 517). Klein's investigations on transformation of the 7th order (Math. Ann., Bd. 13, p. 428) are based on the curve $\psi_3 = z_1^3 z_2 + z_2^3 z_3 + z_3^3 z_1 = 0$ (see also Haskell, *American Journal*, Vol. XIII, p. 1). The five invariants (58) and the two syzygies (57) occur also in a paper by Brioschi, "Ueber die Jacobischen Modulargleichungen vom 8^ten^ Grad" (Math. Ann., Bd. 15, p. 241).

The corresponding values of n and v_n are these:

$n =$	1	2	3	4	5	6	7	8	9	10	11	12
$v_n =$	10	7	4	1	11	8	5	2	12	9	6	3

The complete system consists of $\sqrt[3]{A}$, E, ψ_1, ψ_2, ψ_3, ψ_4. We find

$$\begin{aligned}
\psi_5 &= \psi_2\psi_3 - A^2\psi_4^2 + A^3\psi_3, \\
\psi_6 &= \psi_3^2 - 2A^3\psi_4, \\
\psi_7 &= \psi_3\psi_4 - A\psi_2 - 3A^4, \\
\psi_8 &= \psi_4^2 - 2A\psi_3, \\
\psi_9 &= \psi_3^3 - 3A^3\psi_3\psi_4 + 3A^7, \\
\psi_{10} &= \psi_3^2\psi_4 - A\psi_2\psi_3 - A^3\psi_4^2 - 2A^4\psi_3, \\
\psi_{11} &= \psi_3\psi_4^2 - A\left(\psi_2\psi_4 + \psi_3^2\right) - 2A^4\psi_4, \\
\psi_{12} &= \psi_4^3 - 3A\psi_3\psi_4 + 3A^5.
\end{aligned}$$

The three relations between the six forms of the complete system are

$$\begin{aligned}
\psi_2^2 &= \psi_1\psi_3 - A\psi_4^3 + 5A^2\psi_3\psi_4 - 3A^3\psi_2 - 9A^6, \\
\psi_3^2 &= \psi_2\psi_4 - A\psi_1 + A^3\psi_4, \\
\psi_1\psi_4 - \psi_2\psi_3 &= AE - A^3\psi_3.
\end{aligned}$$

V. $t = 19$.

The solutions of the congruence

$$\omega^2 - \omega + 1 \equiv 0 \pmod{19}$$

are $v = 12$, $w = 8$. The reduced system consists of

$$\sqrt[3]{A},\ E \text{ and } 18 \text{ functions } \psi_n = (y_1^n y_2^{v_n}) \qquad (n = 1, 2, \ldots. 18).$$

Corresponding values of n and v_n are

$n =$	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
$v_n =$	12	5	17	10	3	15	8	1	13	6	18	11	4	16	9	2	14	7

The complete system consists of $\sqrt[3]{A}$, E, ψ_1, ψ_2, ψ_5, ψ_8.

We find

$$\begin{aligned}
\psi_3 &= \psi_1\psi_2 - A\psi_5\psi_8 + A^4\psi_5, \\
\psi_4 &= \psi_2^2 - 2A^2\psi_5, \\
\psi_6 &= \psi_2^3 - 3A^2\psi_2\psi_5 + 3A^7, \\
\psi_7 &= \psi_2\psi_5 - A^2\psi_8 - 3A^5, \\
\psi_9 &= \psi_2^2\psi_5 - 2A^2\psi_5^2 - A^3\psi_1 - A^5\psi_2,
\end{aligned}$$

$$\psi_{10}=\psi_2\psi_8-A\psi_1-A^3\psi_2,$$
$$\psi_{11}=\psi_2^2\psi_5-A^2\psi_2(2\psi_5^2+\psi_2\psi_8)+A^4\psi_5\psi_8-2A^5\psi_2^2+4A^7\psi_5,$$
$$\psi_{12}=\psi_2^2\psi_8-A\psi_1\psi_2-A^2\psi_5\psi_8-2A^5\psi_5,$$
$$\psi_{13}=\psi_5\psi_8-A\psi_2^2+A^3\psi_5,$$
$$\psi_{14}=\psi_2^2\psi_5^2-2A^2\psi_2\psi_5\psi_8+A^4\psi_8^2-6A^5\psi_2\psi_5+4A^7\psi_8+9A^{10},$$
$$\psi_{15}=\psi_2\psi_5\psi_8-A\psi_2^3-A^2\psi_8^2+3A^3\psi_2\psi_5-3A^5\psi_8-6A^8,$$
$$\psi_{16}=\psi_8^2-2A\psi_2\psi_5+2A^3\psi_8+6A^6,$$
$$\psi_{17}=\psi_2^2\psi_5\psi_8-A\psi_1\psi_2\psi_5-A^2\psi_2\psi_8^2+A^3(\psi_1\psi_8-\psi_2^2\psi_5)-2A^5\psi_2\psi_8+2A^6\psi_1+2A^8\psi_2,$$
$$\psi_{18}=\psi_2\psi_8^2-A(\psi_1\psi_8+\psi_2^2\psi_5)+A^3(2\psi_5^2-\psi_2\psi_8)+A^4\psi_1.$$

The three relations between the six forms of the complete system are

$$\psi_1^2=E\psi_2-\psi_5\psi_8^2+4A\psi_2\psi_5^2-3A^3\psi_5\psi_8-3A^4\psi_2^2-9A^6\psi_5,$$
$$\psi_2^3=\psi_1\psi_5-A\psi_8^2+5A^2\psi_2\psi_5-3A^4\psi_8-9A^7,$$
$$\psi_5^2=\psi_2\psi_8-A\psi_1+A^3\psi_2.$$

§5. *Order and Subgroups of G.*

Let us consider the three substitutions

$$T,\quad STS^{-1}=U,\quad\text{and}\quad S^{-1}TS=V,$$

where S and T are the two generating substitutions of G given in (1) and (2):

$$\left.\begin{aligned}z_1'&=a_1z_1\\z_2'&=a_2z_2\\z_3'&=a_3z_3\end{aligned}\right\}=T,\qquad\left.\begin{aligned}z_1'&=a_2z_1\\z_2'&=a_3z_2\\z_3'&=a_1z_3\end{aligned}\right\}=U,\qquad\left.\begin{aligned}z_1'&=a_3z_1\\z_2'&=a_1z_2\\z_3'&=a_2z_3\end{aligned}\right\}=V,$$

and let us find all those substitutions R which are generated by T, U and V. It is obvious that *they are all interchangeable.* On account of $a_1a_2a_3=1$ we have $V=(TU)^{-1}$, therefore we may confine ourselves to T and U. The general form of all possible substitutions R generated by T and U is T^mU^n, and since P is the period of T and of U, the substitutions R are given by this table:

$$\left.\begin{array}{lllll}1 & ,T & ,T^2 & \ldots\ldots T^{P-1} & ,\\ U & ,TU & ,T^2U & \ldots\ldots T^{P-1}U & ,\\ U^2 & ,TU^2 & ,T^2U^2 & \ldots\ldots T^{P-1}U^2 & ,\\ \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots\ldots\ldots & \\ U^{P-1}, & TU^{P-1}, & T^2U^{P-1} & \ldots\ldots T^{P-1}U^{P-1}. & \end{array}\right\}\tag{59}$$

Now it may happen that some power of T will be equal to some power of U. The condition for $T^\lambda=U^\mu$ is:

$$a_1^\lambda=a_2^\mu,\ a_2^\lambda=a_3^\mu,\ a_3^\lambda=a_1^\mu\tag{60}$$

or, see (3),

$$\frac{\lambda k_1}{m_1}-\frac{\mu k_2}{m_2}=\lambda_1,$$
$$\frac{\lambda k_2}{m_2}-\frac{\mu k_3}{m_3}=\lambda_2,$$
$$\frac{\lambda k_3}{m_3}-\frac{\mu k_1}{m_1}=\lambda_3,$$

where $\lambda_1, \lambda_2, \lambda_3$ are three integers. But these three equations coincide exactly with the equations (7) which have already been solved in §1. The solutions of our equations (60) are, therefore, see (26): $\lambda=\alpha$, $\mu=-\beta$, or in full length:

$$\lambda=p_1p_2r\,(lt+n),$$
$$\mu=-p_1p_2r\,(mt+nv).$$

The smallest power λ satisfying the equation $T^\lambda=U^\mu$ is therefore

$$\lambda=p_1p_2r=\vartheta,$$

and the corresponding power $\mu=\vartheta\,(t-v)$; but $v+w-1=t$, hence $t-v=w-1$. We have, therefore,

$$T^\vartheta=U^{\vartheta(w-1)},$$

and likewise

$$U^\vartheta=T^{\vartheta(v-1)}.$$

The consequence is that only the substitutions of the first $\vartheta-1$ rows of table (59) will be all distinct from each other, while the remaining substitutions will be equal to some substitutions of the first $\vartheta-1$ rows. We have then altogether $\vartheta.P$ distinct substitutions R. But every substitution of G can be thrown into the form R, RS or RS^2, because SRS^{-1} is again one of the substitutions R, say R', so that $SR=R'S$. G contains, therefore, $3\vartheta P$ substitutions. Thus we have the result:

The order of G is $3\vartheta P=3\vartheta^2t=3p_1^2p_2^2r^2t$.

If $t=1$, we have $\vartheta=P$, and the order of $G=3P^2$; in this case the P^2 substitutions of table (59) are all distinct. If $\vartheta=1$, we have $t=P$ and the order of $G=3P$ so that the distinct substitutions R will be given by the first row of table (59).

In every case the substitutions R form a *self-conjugate subgroup* H within G of order $\vartheta P=\vartheta^2t$. But there exists another self-conjugate subgroup H' of order t. This is formed by the t powers of T^ϑ, for there is

$$ST^\vartheta S^{-1}=U^\vartheta=T^{\vartheta(v-1)},$$

and

$$RT^\vartheta R^{-1}=T^\vartheta.$$

Evidently H' is contained in H. If $\vartheta=1$, H' coincides with H; and if $t=1$, H' is reduced to unity.

UNIVERSITY OF CHICAGO, *December* 24, 1894.

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On Irrational Covariants of certain Binary Forms.

By E. Study.

In the following pages we intend to study the most important irrational covariants of binary cubics and quartics and of some other special binary forms.

The subject was entered upon by the great originator of the theory of invariants, Cayley, and it has been brought to perfection in some respects, through the application of refined methods, by Clebsch and others. There are, however, quite a number of details which make further investigations desirable, especially when we consider the intimate connection of the subject with some parts of the modern theory of functions. For reasons of this kind it was almost inevitable to the author to work the whole subject over again. Presenting, in a carefully chosen system of notation, the result of this rather laborious task to the mathematical world, I hope that it will be useful to those who have to deal with the numerous applications of the binary quantics of the lowest orders.

Instead of summarizing our new results in detail, we prefer to signalize the points in which our way of looking at the matter differs from the usual one.

As to *algebraical equations*, Cayley's principal point of view seems to have been to derive solutions of the equations of the lowest degrees from the theory of invariants. He wrote an equation $f=0$, say of the third degree, homogeneously, and based the solution upon the identical relation among the covariants of the cubic form f. Thus the introduction of irrational covariants, representing the linear factors of f, was to him chiefly an intermediate step in the investigation of the solution of $f=0$, that is to say, of the *vanishing points* of the quantic f.

While sharing the general views as to the beauty and importance of Cayley's discovery, we venture to profess a different opinion concerning the source of this importance. The idea of solving the equation $f=0$ by means of the theory of invariants is, as we are inclined to think, open to some criticism,[*] even if we do not mind the fact that this solution had been known long before anybody thought of invariants. On the other hand, the linear factors of f are roots of a certain equation $F=0$, containing the variables of f as parameters in their coefficients. This equation is, of course, *not* furnished by the ordinary treatment of the equation $f=0$; and even in the theory of invariants it has not yet received much attention. It seems to us, however, to be the main feature of the theory; we shall therefore endeavor to point out distinctly its very special properties.

We have here a problem in which the theory of equations and the theory of invariants are superposed, or rather we have two different problems:

1st. To decide whether or not algebraic equations $F=0$ exist, the coefficients of which belong to the system of rational invariants and covariants of a given system of quantics $f_1, f_2, \ldots\ldots$, and the roots of which represent divisors of a form f occurring in this system; and, if possible, to determine these equations, especially the "simplest" ones (according to properly chosen definitions of "simplicity").

2d. To solve these equations.

The reader will notice that the first question does not always admit of an affirmative answer, and that also several equally simple solutions may exist, differing by factors of proportionality, and not reducible to one another by rational substitutions.[†] In the actual treatment of special cases it will not be necessary to keep separated the two steps, or to urge the definition of the

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[*] Namely, the covariants do not seem to act any *necessary* part in the solution of a given equation; actually Cayley's form of the solution does not present itself readily in Galois' theory.

To show the difference of the standpoints in a simple example, we may compare Cayley's solution of a quadratic equation, which involves a variable parameter p (or rather a pair of homogeneous parameters $p_1 : p_2$) with the ordinary solution, which is free from this complication. The parameter is accounted for by the fact, that in the system of covariants of a binary quadratic form f no equation $F=0$ exists, the roots of which would be linear factors of f; whereas such an equation can be formed in the simultaneous system of the quadratic form f and a linear form p. The parameter would be utterly superfluous if merely the solution of the equation $f=0$ were in question.

[†] An interesting example is furnished by the combinant of the eighth order in the theory of a binary quartic. We are to decompose this form in no less than *four* different manners into a pair of biquadratic factors, each representing four equianharmonic points. The irrational covariants defining these factors have very different properties. The geometrical interpretation of a quantic by the mere group of its vanishing points fails entirely to give an account of such occurrences.

"simplest" solution of the first question; this definition is furnished implicitly by the investigation itself.

The said equations $F = 0$ contain variable parameters in their coefficients. Now, in all cases dealt with in this paper they enjoy a remarkable property: *they have resolvents containing a smaller number of parameters.*

Since very little is known as to equations with parameters in general, we may claim some interest for our developments, however special they may appear, from the mere standpoint of the theory of equations.

Our notation is the so-called symbolical one, which is by far the most convenient notation at least for special researches of this character. The reader is supposed to be familiar with the methods applied in the standard work of Clebsch, "Theorie der binären Formen" (Leipzig, 1872), especially with the properties of the *rational* covariants of binary cubics and quartics contained in Chapter IV. Instead of Clebsch, also Gordan's "Vorlesungen über Invariantentheorie" (herausgegeben von Kerschensteiner, II, Leipzig, 1887) may be consulted.

It must be mentioned that we differ slightly from both authors in the notation of binary variables. We denote a binary quantic by $(ax)^n = (a_1x_2 - a_2x_1)^n$, instead of writing $a_x^n = (a_1x_1 + a_2x_2)^n$. Thus we avoid the introduction of the so-called principle of contragredience, which has no genuine right of existence in the binary domain.

I.—THE CUBIC.

In the theory of a binary cubic we introduce, for certain reasons which will appear later on, a system of notation slightly different from that used by Clebsch. Thus denoting the forms

$$f,\ \Delta,\ Q,\ R$$

of Clebsch by

$$p,\ 2\delta,\ q,\ 2r,$$

we have the following forms, which constitute what is called the complete system of p:

$$\begin{aligned}
p &= (px)^3 = (p'x)^3 = \ldots\ldots, \\
\delta &= (\delta x)^2 = \tfrac{1}{2}(pp')^2(px)(p'x), \\
q &= (qx)^3 = 2(p\delta)(px)^2(\delta x) = -(pp')^2(pp'')(p'x)(p''x)^2, \\
r &= 2(\delta\delta')^2 = \tfrac{1}{2}(pq)^3.
\end{aligned} \tag{1}$$

These forms are connected by the syzygy

$$rp^2 + 4\delta^3 + q^2 = 0, \tag{2}$$

viz. $$-4\delta^3 = \{q + \sqrt{-r}.p\}\{q - \sqrt{-r}.p\}. \tag{3}$$

Consequently we are able to decompose $-\delta$ into its linear factors (σx) and (τx), defining these forms by means of the equations

$$\begin{gathered}(\sigma x) = \sqrt[3]{\frac{q + \sqrt{-r}.p}{2}}, \quad (\tau x) = \sqrt[3]{\frac{q - \sqrt{-r}.p}{2}}, \\ (\sigma x)(\tau x) = -\delta = -(\delta x)^2.\end{gathered} \tag{4}$$

These forms are *irrational covariants* of f; *they are roots of the sextic equation*

$$y^6 - qy^3 - \delta^3 = 0. \tag{5}$$

Now, from the equations

$$\begin{aligned}(\sigma x)^3 + (\tau x)^3 &= q, \\ (\sigma x)^3 - (\tau x)^3 &= \sqrt{-r}.f,\end{aligned} \quad (\sigma x)(\tau x) = -(\delta x)^2$$

we derive $(\sigma\tau)^3 = r\sqrt{-r}$, $(\sigma\tau)^2 = -r$, that is to say,

$$(\sigma\tau) = -\sqrt{-r}. \tag{6}$$

The linear forms (σx), (τx) thus defined evidently give rise to an unlimited number of irrational covariants of the cubic p contained in the form $\lambda(\sigma x)^n + \mu(\tau x)^n$. Among these forms we find certain linear forms proportional to linear factors of p, and others proportional to those of q, or, as we may say briefly, we find among our irrational covariants the linear factors of p and q.

Let $\sqrt{-3}$ be an arbitrary but definite value of the square root of -3, and denote by ε_1, ε_2, ε_3 the cubic roots of unity, derived from 1, $\varepsilon = \frac{-1+\sqrt{-3}}{2}$, $\varepsilon^2 = \frac{-1-\sqrt{-3}}{2}$ by any *cyclical* permutation, finally by $\bar{\varepsilon}_1$, $\bar{\varepsilon}_2$, $\bar{\varepsilon}_3$ their conjugate values 1, ε^2, ε, arranged in the corresponding order. Then the equations

$$\begin{aligned}(\sigma\tau).(\lambda x) &= \bar{\varepsilon}_1.(\sigma x) - \varepsilon_1.(\tau x), \\ (\sigma\tau).(\mu x) &= \bar{\varepsilon}_2.(\sigma x) - \varepsilon_2.(\tau x), \\ (\sigma\tau).(\nu x) &= \bar{\varepsilon}_3.(\sigma x) - \varepsilon_3.(\tau x),\end{aligned} \tag{7}$$

will define certain irrational covariants of the degree -1, fulfilling the conditions

$$0 = (\lambda x) + (\mu x) + (\nu x), \tag{8}$$

$$\begin{aligned} 3\delta &= r\{(\mu x)(\nu x) + (\nu x)(\lambda x) + (\lambda x)(\mu x)\} \\ &= -\frac{r}{2}.\{(\lambda x)^2 + (\mu x)^2 + (\nu x)^2\}, \end{aligned} \tag{9}$$

$$p = r.(\lambda x)(\mu x)(\nu x) = \frac{r}{3}.\Sigma(\lambda x)^3, \tag{10}$$

$$3\sqrt{-3}.q = r.\sqrt{-r}.\{(\mu x)-(\nu x)\}\{(\nu x)-(\lambda x)\}\{(\lambda x)-(\mu x)\}, \tag{11}$$

$$-\frac{\sqrt{-3}}{(\sigma\tau)} = \frac{\sqrt{-3}}{\sqrt{-r}} = (\mu\nu) = (\nu\lambda) = (\lambda\mu). \tag{12}$$

Hence the forms (λx), (μx), (νx) *are the roots of the cubic equation*

$$rx^3 - * + 3\delta.x - p = 0.^* \tag{13}$$

Their immediate expressions by means of radicals are given by the formulæ

$$\begin{aligned} x &= \sqrt[3]{\frac{rp + q\sqrt{-r}}{2r^3}} + \sqrt[3]{\frac{rp - q\sqrt{-r}}{2r^3}}, \\ &\sqrt[3]{(+)}.\sqrt[3]{(-)} = -r.\delta. \end{aligned} \tag{14}$$

In the same way the decomposition of the cubic covariant q is connected with three irrational covariants $(\lambda' x)$, $(\mu' x)$, $(\nu' x)$ of the degree $+1$.

We may first introduce the notation

$$(\sigma' x) = \frac{(\sigma x)}{(\sigma\tau)}, \quad (\tau' x) = \frac{(\tau x)}{(\tau\sigma)}. \tag{4b}$$

These forms are roots of the sextic equation

$$r^3.y'^6 - r^2 p.y'^3 - \delta^3 = 0. \tag{5b}$$

Further, we define three linear forms $(\lambda' x)$, $(\mu' x)$, $(\nu' x)$ by means of the formulæ

$$(\sigma'\tau').(\lambda' x) = \bar{\varepsilon}_1(\sigma' x) - \varepsilon_1(\tau' x), \text{ etc.} \tag{7b}$$

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[*] This equation is evidently the *simplest* one among all those by which linear forms, proportional to the linear factors of p, can be defined. It has the lowest order possible, namely, *three*, and the degrees of the coefficients do not surpass four. By these properties it is defined, if we do not mind substitutions of the form $x = \rho^{x'}$, where ρ is a rational number. But it is even the simplest equation with respect to the numerical coefficients.

The linear factors of p used by Cayley and Clebsch are roots of an equation of the *sixth* degree. Similar remarks hold in other cases.

or $$-(\lambda' x) = \bar{\varepsilon}_1(\sigma x) + \varepsilon_1(\tau x), \text{ etc.,} \tag{7c}$$

or finally, $$-\sqrt{-3}.(\lambda' x) = (\sigma\tau).\{(\mu x) - (\nu x)\}, \tag{7d}$$

etc., with cyclical permutation of λ, μ, ν.

Then we have

$$0 = (\lambda' x) + (\mu' x) + (\nu' x), \tag{8b}$$

$$\begin{aligned} 3\delta &= (\mu' x)(\nu' x) + (\nu' x)(\lambda' x) + (\lambda' x)(\mu' x) \\ &= -\tfrac{1}{2}\{(\lambda' x)^2 + (\mu' x)^2 + (\nu' x)^2\}, \end{aligned} \tag{9b}$$

$$q = (\lambda' x)(\mu' x)(\nu' x) = \tfrac{1}{3}\Sigma(\lambda' x)^3, \tag{10b}$$

$$3\sqrt{-3}.p = \frac{1}{\sqrt{-r}}\{(\mu' x) - (\nu' x)\}\{(\nu' x) - (\lambda' x)\}\{(\lambda' x) - (\mu' x)\} \tag{11b}$$

$$-\frac{\sqrt{-3}}{(\sigma'\tau')} = -\sqrt{-3}\sqrt{-r} = (\mu'\nu') = (\nu'\lambda') = (\lambda'\mu'). \tag{12b}$$

Hence the forms $(\lambda' x)$, $(\mu' x)$, $(\nu' x)$ *are the roots of the cubic equation*

$$z^3 - * + 3\delta.z - q = 0. \tag{13b}$$

Their expressions in terms of radicals are given by the formulæ

$$\begin{gathered} z = \sqrt[3]{\frac{q - p\sqrt{-r}}{2}} + \sqrt[3]{\frac{q + p\sqrt{-r}}{2}}, \\ \sqrt[3]{(-)}.\sqrt[3]{(+)} = -\delta. \end{gathered} \tag{14b}$$

These formulæ put the reciprocity between the forms p and q into evidence: *Our formulæ are interchanged among each other when we replace*

$$p\,,\quad \delta\,,\qquad q\,,\quad r\,,\qquad (\sigma x),\ (\sigma' x),\ (\lambda x),\qquad (\lambda' x),\ \textit{etc.},$$

respectively by

$$\frac{q}{r},\ \frac{\delta}{r},\ -\frac{p}{r},\ \frac{1}{r},\ -(\sigma' x),\ (\sigma x),\ (\lambda' x),\ -(\lambda x),\ \textit{etc.}^*$$

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[*] $\sqrt{-r}$ has to be replaced by $-\frac{1}{\sqrt{-r}}$.—. It is this law of reciprocity by which the previously mentioned change of notation is suggested. We should not have obtained so simple numerical coefficients in our formulæ if we had operated with the forms Δ and R of Clebsch instead of our δ and r. It is worth noticing that the very same change of notation presents itself still from a quite different point of view. We have avoided the common divisor 2 which appears in the expressions of Δ and R in terms of coefficients of the cubic. (See Clebsch, Binäre Formen, p. 114.)

We recognize in this law of reciprocity the expression of a linear transformation of the binary domain, transforming the vanishing points of p into those of q and *vice versa.*

The equations (5), (5b), (13), (13b) contain a variable parameter $(x_1 : x_2)$ in their coefficients. But they are very special equations of this character. The root is in all cases an *integral* function of the parameter $x_1 : x_2$; in other terms, the Riemann surfaces belonging to the algebraic functions defined by these equations break up into separated leaves. Secondly, these equations have a quadratic resolvent $(\lambda^2 + r = 0)$ which is entirely free from the parameter $(x_1 : x_2)$.

The irrational covariants defined by (5) and (5b) represent in either case the linear factors of δ; but whereas by the solution of (5) δ itself is decomposed into two linear factors, the other equation (5b) decomposes the product $r.\delta$.

In the same way by the solution of (13) not p itself is decomposed into three linear factors, but the product r^2p. The decomposition of p itself into a product of three irrational linear covariants would evidently depend upon the solution of an equation of the *ninth* degree, since the new radical $\sqrt[3]{r}$ has to be introduced. But if the cubic is considered as the cubic covariant of another cubic, which is supposed to be rationally known, then it can be decomposed by the solution of a cubic equation, as is shown by the formula (13b). If we should extend our considerations to the theory of a binary *quintic*, we should reach, in the case of the so-called *canonizant*, another simplification. The vanishing points of this cubic can be represented by irrational covariants of the quintic, which depend upon the roots of a cubic equation whose coefficients are mere invariants (viz. free from the parameter $(x_1 : x_2)$).

Our formulæ contain, of course, also the decomposition of the *sextic* $s^2.rp^2 + t^2.q^2$, wherein $s^2 : t^2$ denotes an arbitrary ratio, into its linear factors.

We have

$$s^2.rp^2 + t^2.q^2 = (t.q + s.\sqrt{-r}.p)(t.q - s.\sqrt{-r}.p). \tag{15}$$

Hence, denoting the two factors of this expression by p_1 and p_2, we have

$$\begin{aligned} p_1 &= t.q + s.\sqrt{-r}.p, \\ \delta_1 &= (t^2 - s^2).r.\delta, \\ q_1 &= (t^2 - s^2).r\sqrt{-r}.(s.q + t\sqrt{-r}.p), \\ r_1 &= (t^2 - s^2)^2.r^3. \end{aligned} \tag{16}$$

The last equation enables us to define

$$\sqrt{-r_1}=(t^2-s^2).r\sqrt{-r}, \tag{16b}$$

whence

$$\begin{aligned}(\sigma_1x)&=\varepsilon_\kappa\sqrt[3]{(s^2-t^2)(s+t)}.\sqrt{-r}.(\sigma x),\\ (\tau_1x)&=\bar{\varepsilon}_\kappa\sqrt[3]{(s^2-t^2)(s-t)}.\sqrt{-r}.(\tau x),\\ (\sigma_1x).(\tau_1x)&=(t^2-s^2).r.(\sigma x).(\tau x).\end{aligned} \tag{17}$$

From these equations we derive the expressions of (λ_1x), etc. Replacing (λ_1x) by $(\bar{\lambda}_1x)=(t^2-s^2).r.(\lambda_1x)$, and denoting the two cube roots in (17) by R and $\bar{R}$, we find

$$(\lambda_1x)=\tfrac{1}{3}\sqrt{-r}.\Big\{(\varepsilon_1R+\bar{\varepsilon}_1\bar{R})(\lambda x)+(\varepsilon_2R+\bar{\varepsilon}_2\bar{R})(\mu x)+(\varepsilon_3R+\bar{\varepsilon}_3\bar{R})(\nu x)\Big\}, \tag{18}$$

etc.; $(\bar{\mu}_1x)$ and $(\bar{\nu}_1x)$ are obtained by means of a cyclical permutation of ε_1, ε_2, ε_3, whereas $(\bar{\lambda}_2x)$ is obtained by interchange of ε_κ and $\bar{\varepsilon}_\kappa$.

The quantities $(\bar{\lambda}_1x)$, $(\bar{\lambda}_2x)$, *etc., are the roots of the sextic equation*

$$\{\bar{x}^3+3(t^2-s^2)\delta.\bar{x}-(t^2-s^2).t.q\}^2+(t^2-s^2)^2.s^2.r.p^2=0. \tag{19}$$

It is a remarkable property of this equation that when the roots of (13) are known, its solution is effected by extraction of radicals *depending merely upon the parameters* s, t. Solving it directly by means of Cardano's formula, we find

$$\begin{aligned}\bar{x}&=\sqrt[3]{(t^2-s^2)(t+s)}.\sqrt[3]{\frac{q\pm\sqrt{-r}.p}{2}}\\ &+\sqrt[3]{(t^2-s^2)(t-s)}.\sqrt[3]{\frac{q\mp\sqrt{-r}.p}{2}}\end{aligned} \tag{20}$$

This is easily seen to be in accordance with (17) and (18), the roots having the values $-R$, $\bar{R}$, (σx), (τx) respectively.

Another remarkable property is shown by the formula (18), namely, the sum of the coefficients of (λx), (μx), (νx), in (18) is zero; and the different roots of (19) are obtained when we exchange these coefficients in all manners possible. We do not insist upon the evident geometrical interpretation of this fact; but we find worth noticing the special form our result assumes when we write the parameters occurring in (15) in a peculiar manner.

Denote a set of three quantities, the sum of which is zero, by e_1, e_2, e_3, and write, as is done in Weierstrass's theory of elliptic functions,

$$\begin{aligned}e_2e_3+e_3e_1+e_1e_2&=-\tfrac{1}{2}(e_1^2+e_2^2+e_3^2)=-\tfrac{1}{4}g_2,\\ e_1e_2e_3&=\tfrac{1}{4}g_3,\\ (e_2-e_3)(e_3-e_1)(e_1-e_2)&=\tfrac{1}{4}\sqrt{g_2^3-27g_3^2}=\sqrt{G}.\end{aligned}$$

Supposing now

$$e_\lambda = \frac{\sqrt{-3}}{6} . (\varepsilon_\lambda R + \bar{\varepsilon}_\lambda \bar{R}),$$

that is to say,

$$(\bar{\lambda}_1 x) = \frac{2\sqrt{-r}}{\sqrt{-3}} . \{e_1(\lambda x) + e_2(\mu x) + e_3(\nu x)\}, \text{ etc.,} \tag{18b}$$

we have

$$t^3 - s^3 = g_2, \; t = 4\frac{\sqrt{G}}{g_3}, \; s = -3\sqrt{-3} . \frac{g_3}{g_2},$$

and the equation (19) is transformed into

$$\{\bar{x}^3 + 3g_2 . \delta . \bar{x} - 4 . \sqrt{G} . q\}^2 - 27g_3^2 . r . p^2 = 0. \tag{19b}$$

By its solution the sextic

$$\begin{aligned} & 16G . q^2 - 27g_3^2 . rp^2 = \\ = & -g_2^3 . rp^2 - 64G . \delta^3 = \\ = & \;\; g_2^3 . q^2 + 108g_3^2 . \delta^3 \end{aligned} \tag{16b}$$

is decomposed into its linear factors. The square root $\sqrt{G}$ can of course be avoided by introducing $\sqrt{G} . \bar{x}$ instead of $\bar{x}$.

II.—THE QUARTIC AND THE OCTAHEDRON.

We shall now make an investigation of certain irrational covariants of a binary quartic f and its sextic covariant t, the so-called octahedron, which we may consider also, as is well known, as an independent sextic F, satisfying the condition $(F, F)_4 = 0$.*

Before doing so, we put together the forms of the complete system of f, and certain relations among these quantics, to be used in our calculations. All these relations can be found among, or easily be derived from, the formulæ developed in the above-mentioned treatise of Clebsch (§40–51, §111), to which we refer for the proofs. As far as the theory of the sextic t is concerned, the reader may consult also Klein's "Icosahedron" (Leipzig, 1884, I, §5, 10, 12) and the literature quoted there.

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* Gordan-Kerschensterner, Binäre Formen, §19. Here and further on we use Gordan's abbreviation $(\psi, \chi)_\kappa$ for the bilinear covariant $(ab)^\kappa (ax)^{n-\kappa} (bx)^{m-\kappa}$ of two binary forms $\psi = (ax)^n$ and $\chi\,(bx)^m$.

§1. *Rational Covariants.*

Dealing with the rational covariants of a binary quartic f, we shall use a system of notation which has already been employed by Brioschi and H. Weber, and which is, for more than one reason, more convenient than the usual system of notation as employed by Clebsch and most of his followers. Thus denoting the forms

$$f,\ H,\ T,\ i\ ,\ j$$

of Clebsch respectively by

$$f,\ 2h,\ t,\ 2g_2,\ 6g_3,$$

we have the following forms, constituting the complete system of rational covariants of f:

$$\begin{aligned} f &= (ax)^4 = (a'x)^4 = \ldots\ldots = (a_1x_2 - a_2x_1)^4 = \ldots\ldots, \\ h &= (hx)^4 = (h'x)^4 = \ldots\ldots = \tfrac{1}{2}(aa')^2(ax)^2(a'x)^2 \quad, \\ t &= (tx)^6 = (t'x)^6 = \ldots\ldots = 2(ah)(ax)^3(hx)^3 \quad, \\ g_2 &= \tfrac{1}{2}(aa')^4,\ g_3 = \tfrac{1}{3}(ah)^4 = \tfrac{1}{6}(aa')^2(aa'')^2(a'a'')^2 \quad. \end{aligned} \qquad (1)$$

These forms are connected by a single syzygy, namely,

$$t^2 + 4h^3 - g_2hf^2 + g_3f^3 = 0. \qquad (2)$$

This most important formula is the clue to the whole theory of irrational covariants.

By the side of the fundamental invariants and covariants we are to consider certain combinations of the above forms. First, of course, the discriminant, which is at the same time an invariant (quasi-discriminant) of t:

$$G = \frac{g_2^3 - 27g_3^2}{16} = \frac{3(t,\ t)_6}{16}. \qquad (3)$$

Further, we introduce a special sign for a certain covariant of the fourth order and fourth degree:

$$\phi = (\phi x)^4 = \tfrac{1}{2}(2g_2.h - 3g_3.f). \qquad (4)$$

It has, among others, these important properties: it is the one form of the pencil $\kappa f + \lambda h$ that is conjugate to f:

$$(f,\ \phi)_4 = (a\phi)^4 = 0. \qquad (5)$$

It is the one form of the pencil $\kappa f + \lambda h$ whose Hessian is proportional to the Hessian of f:

$$\tfrac{1}{2}(\phi,\ \phi)_2 = -\tfrac{1}{3}G.h. \qquad (6)$$

It is a bilinear covariant of f and t:

$$(t, f)_3 = (ta)^3(tx)^3(ax) = (\phi x)^4. \tag{7}$$

It is connected with f, h and t by the identity

$$(t, h)_1 = \tfrac{1}{2} f.\phi. \tag{8}$$

There is, finally, a perfect reciprocity between f and $\dfrac{\sqrt{-3}}{2\sqrt{G}}.\phi$.

Let us consider now the forms of the complete system of the sextic t, which may, when treated as an independent form satisfying the relation $(t, t)_4 = 0$, also be denoted by the sign T.

The forms in question are

$$\begin{aligned} T &= t = (tx)^6, \\ \Phi &= \tfrac{1}{2}(T, T)_2 = \tfrac{1}{2}(tt')^2(tx)^4(t'x)^4, \\ \Psi &= 2(T, \Phi)_1 = -(tt')^2(tt'')(tx)^3(t'x)^4(t''x)^5, \\ R &= \tfrac{1}{36}(T, T)_6 = \tfrac{1}{36}(tt')^6 = \tfrac{4}{27} G. \end{aligned} \tag{9}$$

They, too, are connected by a syzygy:

$$RT^4 + 4\Phi^3 + \Psi^2 = 0.^* \tag{10}$$

The expressions of Φ and Ψ in terms of f and h are

$$\Phi = -\frac{g_2}{3}.h^2 + g_3.hf - \frac{g_2^2}{36}.f^2. \tag{11}$$

$$\Psi = 2g_3.h^3 - \frac{g_2^2}{3}.h^2f + \frac{g_2g_3}{2}.hf^2 + \left(\frac{g_2^3}{108} - \frac{g_3^2}{2}\right)f^3. \tag{12}$$

To these identities we may add the formula

$$9g_2.\Phi + 3\phi^2 + 4G.f^2 = 0, \tag{13}$$

containing another important property of the form ϕ.

§2. *The Irrational Invariants* e_λ, e_μ, e_ν.

Denoting by $t_{\kappa\lambda}$ and $G_{\kappa\lambda}$ the covariant t and the invariant G, derived from the form $\kappa f + \lambda h$ instead of f, and defining a homogeneous cubic function $\Omega(\kappa, \lambda)$ by the formula

$$4\Omega = 4\kappa^3 - g_2\kappa\lambda^2 - g_3\lambda^3, \tag{14}$$

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[*] See Clebsch, Binäre Formen, §111. Our considerations in §5 contain also a proof of this important identity.

we find, following Cayley and Clebsch, that

$$t_{\kappa\lambda} = \Omega . t, \quad G_{\kappa\lambda} = \Omega^2 . G. \tag{15}$$

This shows first that t and G are so-called *combinants* of the pencil $\kappa f + \lambda h$; second, that this pencil contains three forms of vanishing discriminant, which are, on account of the identical vanishing of $t_{\kappa\lambda}$, moreover, perfect squares.

These forms are given by the expressions

$$h + e_\lambda f, \quad h + e_\mu f, \quad h + e_\nu f,$$

if we understand e_λ, e_μ, e_ν to be the roots of the cubic equation

$$0 = 4e^3 - g_2 e - g_3. \tag{16}$$

The cubic resolvent of this equation, the roots of which are the differences $e_\mu - e_\nu$, $e_\nu - e_\lambda$, $e_\lambda - e_\mu$, is

$$0 = e'^3 + \tfrac{3}{4} g_2 . e' - \sqrt{G}. \tag{17}$$

The sign of the quantity $\sqrt{G}$ may, once for all, be explained by the formula

$$\sqrt{G} = (e_\mu - e_\nu)(e_\nu - e_\lambda)(e_\lambda - e_\mu). \tag{18}$$

e_λ, e_μ, e_ν are irrational covariants of f of the degree one, but they are not combinants of the pencil $\kappa f + \lambda h$.

Comparing (2) with (16), we notice (with Cayley) that

$$-t^2 = 4(h + e_\lambda f)(h + e_\mu f)(h + e_\nu f). \tag{19}$$

Hence we may decompose $\frac{t}{2}$ into a product of three quadratic forms l', m', n', denoting preliminarily by l' or $(l'x)^2$ the square root

$$\sqrt{-h - e_\lambda f}, \text{ (etc.)}.$$

These forms l', m', n' are irrational covariants of f, but they are not covariants of t; that is to say, they are not combinants of the pencil $\kappa f + \lambda h$. *There are, however, certain quadratic forms l, m, n, proportional to l', m', n', which enjoy these important properties.*

In order to find them, we put $l' = r_\lambda . l$, $m' = r_\mu . m$, $n' = r_\nu . n$, and try to determine the multipliers r so as to make the invariants of the forms l, m, n *numerical* and equal to each other. The forms thus obtained will be combinants.

We have evidently

$$(m'^2,\ n'^2)_1 = m'.n'.(m',\ n')_1,$$

and

$$(m'^2,\ n'^2)_1 = \tfrac{1}{2}(e_\mu - e_\nu).t = (e_\mu - e_\nu).l'm'n'.$$

$$\therefore\ r_\mu r_\nu.(m,\ n)_1 = r_\lambda.(e_\mu - e_\nu).l.$$

Hence, choosing

$$r_\lambda = -\sqrt{e_\nu - e_\lambda}\sqrt{e_\lambda - e_\mu},\ \ r_\mu = -\sqrt{e_\lambda - e_\mu}\sqrt{e_\mu - e_\nu},\ \ r_\nu = -\sqrt{e_\mu - e_\nu}\sqrt{e_\lambda - e_\mu},$$

we obtain a system of three forms l, m, n, which are covariants of t as well as of f. Their degree, in both cases, is *zero*.

These forms are the true central point in the theory of the most important irrational covariants of f and t; we therefore enter upon a more careful study of them.

§3. *The Quadratic Forms* l, m, n.

The quadratic forms $(lx)^2$, $(mx)^2$, $(nx)^2$ found in §2 fulfil, as a simple calculation will show, the following system of conditions:

$$\begin{aligned}(m,\ n)_1 &= (mn)(mx)(nx) = -(lx)^2,\\ (n,\ l)_1 &= (nl)(nx)(lx) \ \ = -(mx)^2,\\ (l,\ m)_1 &= (lm)(lx)(mx) \ \ = -(nx)^2,\end{aligned} \tag{20}$$

$$\begin{aligned}\tfrac{1}{2}(ll')^2 = \tfrac{1}{2}(mm')^2 = \tfrac{1}{2}(nn')^2 &= 1,\\ \tfrac{1}{2}(mn)(nl)(lm) &= 1,\\ (mn)^2 = (nl)^2 = (lm)^2 &= 0,\end{aligned} \tag{21}$$

from which we may derive the further formulæ

$$l^2 + m^2 + n^2 = 0, \tag{22}$$

$$\begin{aligned}(m^2,\ n^2)_1 &= -mnl, & (m^2,\ n^2) &= \tfrac{2}{3}l^2,\\ (m^2,\ n^2)_3 &= 0\ \ , & (m^2,\ n^2)_4 &= -\tfrac{4}{3},\\ (l^2,\ l^2)_2 &= \tfrac{2}{3}l^2\ \ , & (l^2,\ l^2)_4 &= \tfrac{8}{3},\ \text{etc.}\end{aligned} \tag{23}$$

By the side of these forms l, m, n containing the variables x_1, x_2, we may consider their polars $(lx)(ly)$, etc., containing two sets of variables x, y. First we notice that

$$(lx)^2.(ly)^2 = (lx)(ly).(lx)(ly) + (xy)^2. \tag{24}$$

Further we may establish the following theorem, which contains a very important property of the forms l, m, n:

Denoting by s, x, y, z any four points (sets of binary variables), the quadrilinear forms

$$\begin{aligned} \mathfrak{X}_0 &= (sx).(yz) \;, \\ \mathfrak{X}_1 &= (ls)(lx).(ly)(lz) \;, \\ \mathfrak{X}_2 &= (ms)(mx).(my)(mz), \\ \mathfrak{X}_3 &= (ns)(nx).(ny)(nz) \;, \end{aligned} \tag{25}$$

and the corresponding forms $\mathfrak{Y}_\kappa$, $\mathfrak{Z}_\kappa$, derived from $\mathfrak{X}_\kappa$ by cyclical permutation of x, y, z, are connected by the linear equations

$$0 =$$

$$\begin{array}{llll} \mathfrak{X}_0 + \mathfrak{Y}_0 + \mathfrak{Z}_0, & \mathfrak{X}_1 + \mathfrak{Y}_2 + \mathfrak{Z}_3, & \mathfrak{X}_2 + \mathfrak{Y}_3 + \mathfrak{Z}_1, & \mathfrak{X}_3 + \mathfrak{Y}_1 + \mathfrak{Z}_2, \\ \mathfrak{X}_1 + \mathfrak{Y}_3 + \mathfrak{Z}_2, & \mathfrak{X}_0 - \mathfrak{Y}_1 + \mathfrak{Z}_1, & \mathfrak{X}_3 + \mathfrak{Y}_0 - \mathfrak{Z}_3, & -\mathfrak{X}_2 + \mathfrak{Y}_2 + \mathfrak{Z}_0, \\ \mathfrak{X}_2 + \mathfrak{Y}_1 + \mathfrak{Z}_3, & -\mathfrak{X}_3 + \mathfrak{Y}_3 + \mathfrak{Z}_0, & \mathfrak{X}_0 - \mathfrak{Y}_2 + \mathfrak{Z}_2, & \mathfrak{X}_1 + \mathfrak{Y}_0 - \mathfrak{Z}_1, \\ \mathfrak{X}_3 + \mathfrak{Y}_2 + \mathfrak{Z}_1, & \mathfrak{X}_2 + \mathfrak{Y}_0 - \mathfrak{Z}_2, & -\mathfrak{X}_1 + \mathfrak{Y}_1 + \mathfrak{Z}_0, & \mathfrak{X}_0 - \mathfrak{Y}_3 + \mathfrak{Z}_3. \end{array} \tag{26}$$

The first of these is the elementary identity

$$(sx)(yz) + (sy)(zx) + (sz)(xy) = 0\,;$$

at the others we arrive by properly chosen processes of polarization with respect to the quadratic forms l, m, n.*

Another no less important property of the forms l, m, n is:

The products $(mx)(my).(nx)(ny)$, $(nx)(ny).(lx)(ly)$, $(lx)(ly).(mx)(my)$ *as well as the product* $(lx)(ly).(mx)(my).(nx)(ny)$ *are polars*, that is to say, the application of the operator $\frac{\partial^2}{\partial x_1 \partial y_2} - \frac{\partial^2}{\partial x_2 \partial y_1}$ to these forms produces zero.

Defining the "composition" or "symbolical multiplication" of two bilinear forms $s = (\alpha x)(\beta y)$ and $s' = (\alpha' x)(\beta' y)$ by the formula $ss' = (\alpha x)(\alpha' \beta)(\beta' y)$ (preferable, in some respects, to the definition $ss' = (\alpha x)(\beta \alpha')(\beta' y)$), the manifoldness of all binary bilinear forms constitutes what is termed a system of complex numbers. Choosing as fundamental units the forms

$$\iota_0 = (xy) \quad, \; \iota_1 = (lx)(ly) \;,$$
$$\iota_2 = (mx)(my), \; \iota_3 = (nx)(ny),$$

we obtain a well-known system of formulæ: *The law of composition of our bilinear forms is identical with the multiplication rule of quaternions.*

The equations $\iota_\kappa = 0$ represent a group of four commutative linear transformations of the binary domain, which interchange, as we shall see in §12 more in detail, the vanishing points of each form of the pencil $\varkappa f + \lambda h$ among each other.*

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* Compare the author's paper, Am. Journal, Vol. XVI (p. 156), No. 12–14.

This explains the symmetry among the four bilinear forms (xy), $(lx)(ly)$, $(mx)(my)$, $(nx)(ny)$, which appears in the first theorem of this section, and which we shall meet again in some other theorems.

§4. *Expressions of the Forms f, h, ϕ, t in Terms of l, m, n.*

The forms l^2, m^2, n^2 belong, according to their definition, to the pencil $\varkappa f + \lambda h$. We find them connected with the forms f and h by the identities:

$$\begin{aligned} l^2 &= \frac{h + e_\lambda f}{(e_\mu - e_\lambda)(e_\nu - e_\lambda)} = \\ &= -\frac{(e_\mu - e_\nu)\,h + e_\lambda(e_\mu - e_\nu)f}{(e_\mu - e_\nu)(e_\nu - e_\lambda)(e_\lambda - e_\mu)} = \frac{1}{\sqrt{G}} \cdot \left\{(e_\mu^2 - e_\nu^2)f - (e_\mu - e_\nu)\,h\right\}, \end{aligned} \tag{27}$$

etc., or, in a different arrangement of the formulæ, and with a slight generalization,

$$(lx)^2.(ly)^2 + \quad (mx)^2.(my)^2 + \quad (nx)^2.(ny)^2 = 2\,(xy)^2, \tag{28}$$

$$\begin{aligned} &e_\lambda.(lx)^2.(ly)^2 + e_\mu.(mx)^2.(my)^2 + e_\nu.(nx)^2.(ny)^2 = (ax)^2(ay)^2, \\ &e_\lambda^2.(lx)^2.(ly)^2 + e_\mu^2.(mx)^2.(my)^2 + e_\nu^2.(nx)^2.(ny)^2 = \\ &\qquad = (ax)^2\,\frac{(aa')^2}{2}\,(a'y)^2 = (hx)^2(hy)^2 + \tfrac{1}{3}\,g_2.(xy)^2. \end{aligned} \tag{29}$$

To these formulæ we add the analogous expressions:

$$\begin{aligned} &\sum(e_\mu - e_\nu).(lx)^2.(ly)^2 = -\frac{3}{2\sqrt{G}}\cdot(\phi x)^2(\phi y)^2, \\ &\sum(e_\mu - e_\nu)^2.(lx)^2.(ly)^2 = \quad \frac{9}{4G}\cdot(\phi x)^2\,\frac{(\phi\phi')^2}{2}\,(\phi y)^2 = \\ &\qquad = -3\{(hx)^2(hy)^2 - \tfrac{1}{3}\,g_2.(xy)^2\}. \end{aligned} \tag{30}$$

Instead of the expression of t, too, we may write down at once the expressions of some of its polars, to be used in our calculations,

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* Compare Cayley, Math. Annalen, t. XV, p. 288; Stephanos, ib. Vol. XXII, p. 299, and the author's paper, Berichte der Königl. Sächs. Gesellschaft der Wissenschaften, 1889, p. 177, §10.

$$\begin{aligned}&(lx)(ly)\,.\,(mx)(my)\,.\,(nx)(ny)=\\ &=-\frac{[(mx)^2.(ny)^2]^3-[(my)^2.(nx)^2]^3}{4\,(xy)}=\text{etc.}=-\frac{(tx)^3(ty)^3}{2\sqrt{G}},\end{aligned}\tag{31}$$

$$\frac{(lx)^2.(mx)^2.(nx)^2}{3}\cdot\left\{\frac{(ly)^2}{(lx)^2}+\frac{(my)^2}{(mx)^2}+\frac{(ny)^2}{(nx)^2}\right\}=-\frac{(tx)^4(ty)^2}{2\sqrt{G}}\,.\tag{32}$$

Combining (31) with the well known identity

$$2\,(xy)\,.\,(tx)^3(ty)^3=(ax)^4.(hy)^4-(ay)^4.(hx)^4,\tag{33}$$

we have finally

$$(xy)\,.\,(lx)(ly)\,.\,(mx)(my)\,.\,(nx)(ny)=\frac{(ax)^4(hy)^4-(ay)^4(hx)^4}{-4\sqrt{G}}\,.\tag{34}$$

The invariants e_λ, e_μ, e_ν being known, the forms l^2, m^2, n^2 are perfectly determined by the formulæ (27), whereas the forms l, m, n themselves are certain square roots of l^2, m^2, n^2.

There are, *under the said condition*, altogether *four* systems of values of l, m, n satisfying the relations established in §3, our formulæ permitting no more than a simultaneous change of sign to two of the quantities l, m, n.

§5. *The Quadratic Forms l, m, n considered as Covariants of the Sextic T.*

The forms l, m, n are, as we have stated already, combinants of the pencil $\varkappa f+\lambda h$, or covariants of the sextic $t=T$. We have, according to (31),

$$T^2=4G.l^2.m^2.n^2=\tfrac{4}{3}\,G.\{l^6+m^6+n^6\},\tag{35}$$

$$\begin{aligned}\Phi&=\tfrac{1}{2}(T,\,T)_2=\tfrac{4}{3}\,G.\{m^2n^2+n^2l^2+l^2m^2\}=\\ &=-\tfrac{2}{3}\,G.\{l^4+m^4+n^4\},\end{aligned}\tag{36}$$

$$\Psi=2\,(T,\,\Phi)_1=\tfrac{8}{27}\,G\sqrt{G}\,.\,\{m^2-n^2\}\{n^2-l^2\}\{l^2-m^2\},\tag{37}$$

$$R=\tfrac{1}{36}\,(T,\,T)_6=\tfrac{4}{27}\,G.\tag{9}$$

Consequently the biquadratic forms l^2, m^2, n^2 are the roots of the cubic equation

$$R\,.\,X^3-*+\tfrac{1}{3}\Phi.X-\tfrac{1}{27}\,T^2=0.\tag{38}$$

The solution of this equation in terms of radicals is facilitated by the existence of a quadratic resolvent without parameter. On account of the identity (10) we find

$$\begin{aligned}&3X=\sqrt[3]{\frac{RT^2+\Psi\sqrt{-R}}{2}}+\sqrt[3]{\frac{RT^2-\Psi\sqrt{-R}}{2}}\,,\\ &R.\sqrt[3]{(+)}\sqrt[3]{(-)}=-\Phi,\ \sqrt{-R}=\tfrac{2}{9}\sqrt{-3}.\sqrt{G}.\end{aligned}\tag{39}$$

The simplification of the solution of (38) contained in this formula is all that can be reached as long as we operate in the system of the sextic T. But if such a form T is the sextic covariant of a quartic f, which is supposed to be rationally known, then our cubic equation has a cubic resolvent (No. 16) which is free from the parameter $(x_1:x_2)$.*

The quadratic forms l, m, n, defined by (38) and by the equations in §3, are capable of *eight* systems of values, the relations among the forms l, m, n not being changed when we replace l, m, n by $-l, n, m$. Their ambiguity is reduced to *four* after having disposed of the sign of $\sqrt{G}$ (see No. 31), and so it remains after adjunction of e_λ, e_μ, e_ν.

§6. *Other Irrational Covariants of the Sextic T.*

The formulæ hitherto developed give rise to an unlimited number of irrational covariants of T, combinants of the pencil $\varkappa f + \lambda h$. We shall confine ourselves to a consideration of some of the most important ones.

There are first the two *equianharmonic forms* of the pencil $\varkappa f + \lambda h$, or, as we prefer to say in the present connection, the two quartics that are *apolar* with respect to T† and at the same time equianharmonic. There exists no quadratic equation with coefficients belonging to the system of T, the roots of which would be equianharmonic forms of the said pencil; but there are two different quadratic equations whose roots are the *cubes* of such forms.

We denote again, as we did in the theory of the cubic (p. 171), by $\varepsilon_1, \varepsilon_2, \varepsilon_3$ any cyclical permutation of the three cube roots of unity $\varepsilon^0, \varepsilon^1, \varepsilon^2$, and by $\bar{\varepsilon}_1, \bar{\varepsilon}_2, \bar{\varepsilon}_3$ their conjugate values, arranged in the corresponding order. Then the two quartics

$$\begin{aligned} \mathfrak{S} &= -\sqrt{-R}.\{\varepsilon_1 l^2 + \varepsilon_2 m^2 + \varepsilon_3 n^2\}, \\ \mathfrak{T} &= \sqrt{-R}.\{\bar{\varepsilon}_1 l^2 + \bar{\varepsilon}_2 m^2 + \bar{\varepsilon}_3 n^2\}, \end{aligned} \tag{40}$$

as well as the two quartics

$$\begin{aligned} \mathfrak{S}' &= \varepsilon_1 l^2 + \varepsilon_2 m^2 + \varepsilon_3 n^2, \\ \mathfrak{T}' &= \bar{\varepsilon}_1 l^2 + \bar{\varepsilon}_2 m^2 + \bar{\varepsilon}_3 n^2 \end{aligned} \tag{40b}$$

represent the equianharmonic forms in question.

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* The equation (38) has occasionally been mentioned in the author's book, "Methoden zur Theorie der ternären Formen," in order to exemplify the notion of irrational covariant. Unfortunately the equation communicated there has erroneous numerical coefficients.

† The "polars" of f and h with respect to t, viz. the forms $(f, t)_4$ and $(h, t)_4$, vanish, as is well known, identically. Consequently the pencil $\varkappa f + \lambda h$ is made up of the quartics which are "apolar" with respect to t.

After a short calculation we find

$$\mathfrak{S}^3 = \frac{\Psi + T^2\sqrt{-R}}{2},\ \mathfrak{T}^3 = \frac{\Psi - T^2\sqrt{-R}}{2},$$
$$\mathfrak{S}.\mathfrak{T} = \tfrac{3}{8} R.(l^4 + m^4 + n^4) = -\Phi, \tag{41}$$

$$\mathfrak{S}'^3 = \frac{RT^3 - \Psi\sqrt{-R}}{2R^3},\ \mathfrak{T}'^3 = \frac{RT^2 + \Psi\sqrt{-R}}{2R^2},$$
$$\mathfrak{S}'.\mathfrak{T}' = \tfrac{3}{8}(l^4 + m^4 + n^4) = -\frac{\Phi}{R}. \tag{41b}$$

Hence the forms $\mathfrak{S}$, $\mathfrak{T}$ *are roots of the sextic equation*

$$Y^6 - \Psi Y^3 - \Phi^3 = 0, \tag{42}$$

whereas $\mathfrak{S}'$, $\mathfrak{T}'$ *are roots of the sextic equation*

$$R^3 Y'^6 - R^2.T^2.Y'^3 - \Phi^3 = 0. \tag{42b}$$

Postponing the further consideration of the equianharmonic forms of the pencil $\varkappa f + \lambda h$, we may investigate now the *three harmonic forms* of the said pencil.

Introducing the notation

$$L = \tfrac{3}{8}\sqrt{G}\,(m + n)(m - n) = 2e_\lambda.h + \left(\frac{g_2}{3} - 2e_\lambda^2\right)f,$$
$$M = \tfrac{3}{8}\sqrt{G}\,(n + l)\ (n - l) = 2e_\mu.h + \left(\frac{g_2}{3} - 2e_\mu^2\right)f, \tag{43}$$
$$N = \tfrac{3}{8}\sqrt{G}\,(l + m)\,(l - m) = 2e_\nu.h + \left(\frac{g_2}{3} - 2e_\nu^2\right)f,$$

we have the three forms in question, L, M, N being irrational covariants of T of the degree one.

These forms are the roots of the cubic equation

$$Z^3 - * + 3\Phi Z - \Psi = 0, \tag{44}$$

the solution of which in terms of radicals is contained in the formulæ

$$Z = \sqrt[3]{\frac{\Psi + T^2\sqrt{-R}}{2}} + \sqrt[3]{\frac{\Psi - T^2\sqrt{-R}}{2}},$$
$$\sqrt[3]{(+)}.\sqrt[3]{(-)} = -\Phi. \tag{45}$$

The mutual relations of the equations (38), (42), (42b) and (45) are obvious, and need no further explanation; we may add, however, the remark that while (44) gives immediately the decomposition of Ψ into its three conjugate factors, by means of the solution of (38) only the product RT^2 is decomposed.

There exists still an interesting system of *four quadratic forms*, each representing certain two vanishing points of Φ, namely, a pair of vanishing points x, y, connected by the relations

$$(tx)^5(ty) = 0, \quad (ty)^5(tx) = 0, \quad (tx)^4(ty)^2 = 0, \quad (ty)^4(tx)^2 = 0.$$

These forms

$$\begin{aligned} W_0 &= \tfrac{2}{3}\sqrt{G}(l + m + n), \\ W_\lambda &= \tfrac{2}{3}\sqrt{G}(l - m - n), \\ W_\mu &= \tfrac{2}{3}\sqrt{G}(-l + m - n), \\ W_\nu &= \tfrac{2}{3}\sqrt{G}(-l - m + n) \end{aligned} \tag{46}$$

are the roots of the biquadratic equation

$$W^4 + 8RT.W + 12R\Phi = 0,$$

which has, according to its origin, (38) and (44) as cubic resolvents.

The sum of these forms is zero as well as the sum of their squares; their Jacobians are proportional to the quadratic forms occurring in (43).*

§7. *The Equianharmonic Forms of the Pencil $\varkappa f + \lambda h$.*

While apparently enough has been said as to the determination of the forms l^2, m^2, n^2, and of the differences $m^2 - n^2$, $n^2 - l^2$, $l^2 - m^2$ as irrational covariants of f, the pair of equianharmonic forms in the pencil $\varkappa f + \lambda h$ deserves further attention.

We have already found two different pairs of irrational covariants proportional to the forms in question; both are roots of equations of the 6th degree and *covariants of T*. Operating, however, not in the system of T but in the system of the quartic f, we obtain a simpler solution of the problem, to find the equianharmonic forms of the pencil $\varkappa f + \lambda h$. Now these forms can be represented in two essentially different ways, by the roots of a *quadratic* equation, which defines them as covariants of f (but not as covariants of T). We have simply to decompose the product $g_2\Phi$ into two factors of the form $\alpha f + \beta\phi$, using the identity (13).

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*Compare Brill, Math. Annalen, v. 20. Gordan u. Kerschensterner, Binäre Formen (Leipzig, 1887), No. 142.

Let us define these factors $\mathfrak{Y}_+$ and $\mathfrak{Y}_-$ by

$$\begin{aligned} 2\mathfrak{Y}_{\pm} &= 2g_2.h - (3g_3 \pm \tfrac{4}{3}\sqrt{-3}\sqrt{G})f = \\ &= \pm\tfrac{4}{3}\sqrt{-3}\sqrt{G}.\left\{f \pm \frac{\sqrt{-3}}{2\sqrt{G}}\phi\right\} = \\ &= \sum e_\lambda\{2g_2e_\lambda - 3g_3 \mp \tfrac{4}{3}\sqrt{-3}\sqrt{G}\}.l^2 = \\ &= -\tfrac{4}{3}\sqrt{G}.\sum\{e_\mu - e_\nu \pm \sqrt{-3}.e_\lambda\}l^2. \end{aligned} \tag{48}$$

These forms $\mathfrak{Y}_+$, $\mathfrak{Y}_-$ *are the roots of the quadratic equation*

$$\mathfrak{Y}^2 - 2\phi.\mathfrak{Y} - 3g_2.\Phi = 0. \tag{49}$$

The forms $\mathfrak{Y}_+$, $\mathfrak{S}$, $\mathfrak{S}'$, and in the same way the forms $\mathfrak{Y}_-$, $\mathfrak{T}$, $\mathfrak{T}'$, have the same vanishing points; their ratios are therefore (irrational) invariants of f:

$$\begin{aligned} \mathfrak{Y}_+ &= -\sqrt{-3}.\sqrt[3]{4\sqrt{G} - 3g_3\sqrt{-3}}.\mathfrak{S}, \\ \mathfrak{Y}_- &= \sqrt{-3}.\sqrt[3]{4\sqrt{G} + 3g_3\sqrt{-3}}.\mathfrak{T}, \end{aligned} \tag{50}$$

where

$$\begin{gathered} \sqrt[3]{4\sqrt{G} - 3g_3\sqrt{-3}} = \\ = \frac{e_\mu - e_\nu + e_\lambda\sqrt{-3}}{\varepsilon_1} = \frac{e_\nu - e_\lambda + e_\mu\sqrt{-3}}{\varepsilon_2} = \frac{e_\lambda - e_\mu + e_\nu\sqrt{-3}}{\varepsilon_3}, \\ \sqrt[3]{4\sqrt{G} + 3g_3\sqrt{-3}} = \\ = \frac{e_\mu - e_\nu - e_\lambda\sqrt{-3}}{\bar{\varepsilon}_1} = \frac{e_\nu - e_\lambda - e_\mu\sqrt{-3}}{\bar{\varepsilon}_2} = \frac{e_\lambda - e_\mu - e_\nu\sqrt{-3}}{\bar{\varepsilon}_3}. \end{gathered} \tag{51}$$

These formulæ contain the expressions of $\mathfrak{S}$, $\mathfrak{T}$, $\mathfrak{S}'$, $\mathfrak{T}'$ in the domain of rationality of the quantities e_λ, e_μ, e_ν, $\sqrt{-3}$.

Substituting finally instead of $\mathfrak{Y}_+$, $\mathfrak{Y}_-$, the quantities

$$\mathfrak{Y}'_+ = -\frac{\mathfrak{Y}_+}{\sqrt{-3}\sqrt{G}}, \quad \mathfrak{Y}'_- = \frac{\mathfrak{Y}_-}{\sqrt{-3}\sqrt{G}}, \tag{48b}$$

we find, instead of (49), the equation

$$G.\mathfrak{Y}'^2 - \tfrac{4}{3}G.f.\mathfrak{Y}' - g_2.\Phi = 0, \tag{49b}$$

which defines another set of equianharmonic forms, irrational covariants of f. The place of the covariant ϕ, occurring in (49), is now taken by f itself.

§8. *Connection between Cubic and Quartic.*

It is well known that by interpreting the parameter $\varkappa:\lambda$ of the pencil $\varkappa f+\lambda h$ as a point in the binary domain, we obtain the simple connection between the theory of a quartic and that of a cubic (Clebsch, B. F., §41 and 44). Now from our theory of irrational covariants we deduce a new and simple analytical expression of this connection.

With a change of notation, we denote a point of the said binary domain by ξ, and, referring to our theory of cubics (I, No. 7–12), write*

$$(\lambda\xi)\cong e_\mu-e_\nu,\ (\mu\xi)\cong e_\nu-e_\lambda,\ (\nu\xi)\cong e_\lambda-e_\mu, \tag{52}$$

establishing in this way a *one-to-one correspondence* between the binary domain (ξ) and the binary domain the elements of which are the forms of the pencil $\varkappa f+\lambda h$. It follows immediately that

$$\frac{(\lambda'\xi)}{\sqrt{-3}\sqrt{-r}}\cong e_\lambda,\ \frac{(\mu'\xi)}{\sqrt{-3}\sqrt{-r}}\cong e_\mu,\ \frac{(\nu'\xi)}{\sqrt{-3}\sqrt{-r}}\cong e_\nu, \tag{52b}$$

$$\frac{\delta}{r}\cong-\frac{g_2}{4},\ \frac{p}{r}\cong\sqrt{G},\ \frac{q}{\sqrt{-3}.r\sqrt{-r}}\cong\frac{3g_3}{4}. \tag{53}$$

By means of these formulæ, to the vanishing points of the forms l^2, m^2, n^2, and to those of the equianharmonic and harmonic forms of the pencil $\varkappa f+\lambda h$, correspond the vanishing points of a binary cubic and of its quadratic and cubic covariant.

This remarkable theorem which is (save the above analytical expression) due to Clebsch, as it seems, can be generalized in two different ways.

Writing s instead of the variable x in the theory of the cubic, and forming the product $(ps)^3.(\xi s)$, *we have in our formulæ an analytical expression for a transformation of the binary domain by which the pencil $\varkappa f+\lambda h$ is transformed into another pencil of binary quartics, namely, into a pencil of quartics which have three linear factors (λs), (μs), (νs) in common, while the last factor (ξs) alone is variable.* Now forming the invariants g_2' and g_3' of the quartic $\frac{1}{r}(ps)^3.(\xi s)$, we find

$$g_2'\cong\frac{3}{16}r.g_2,\quad g_3'\cong-\frac{3\sqrt{-3}}{64}r\sqrt{-r}.g_3. \tag{54}$$

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* When establishing a principle of transference like the one above, containing a comparison of two theories of different origin, we may use the sign $\cong$ instead of the ordinary sign of equality $=$.

In the following conclusions we assume that the sign $\sqrt{-3}$ has the same meaning in both theories.

Consequently the absolute invariant $\frac{g_2'^3}{g_3'^2}$ *of the product* $(ps)^3.(\xi s)$ *is identical with the absolute invariant* $\frac{g_2^3}{g_3^2}$ *of the quartic* f.

Or, expressing the same result more geometrically:

If we denote the vanishing points of f, *in a properly chosen arrangement, by* r_0, r_λ, r_μ, r_ν, *and let, by means of a linear transformation, the points* r_λ, r_μ, r_ν *coincide with the vanishing points* λ, μ, ν *of the cubic* p; *the point* r_0 *coincides with the point* ξ, *corresponding to our quartic by means of the formulæ* (52).

Later on, when decomposing f into its linear factors, we shall introduce four linear forms (r_0x), $(r_\lambda x)$, $(r_\mu x)$, $(r_\nu x)$, the vanishing points of which represent exactly the arrangement in question.—

Secondly, we may consider more closely the one-to-four correspondence between the points ξ of the binary domain of the cubic p and the points x of the domain of the quartic f, by means of which the point ξ is transformed into the four vanishing points of f, or more generally (considering ξ as variable), into the vanishing points of a form $\varkappa f + \lambda h$.

We may express the transformation in question by means of an equation which contains variables of either domain, (ξ) or (x), and which is obtained by substituting the above values of e_λ, e_μ, e_ν in the expression (29) of f:

$$0 = (\lambda'\xi)\, l^2 + (\mu'\xi)\, m^2 + (\nu'\xi)\, n^2 \cong \sqrt{-3}\,\sqrt{-r}\,.(ax)^4. \tag{55}$$

Now, let in this way to the points $\xi, \eta, \zeta, \ldots.$ correspond the forms $(ax)^4$, $(bx)^4$, $(cx)^4$ of our pencil. We find, by means of the formulæ (32) and (9):

$$\begin{aligned} (\sqrt{-3}\,\sqrt{-r})^2.(ab)^4 &\cong -24\,(\delta\xi)(\delta\eta), \\ (\sqrt{-3}\,\sqrt{-r})^3.(ab)^2(ac)^2(bc)^2 &\cong \quad 24\,(q\xi)(q\eta)(q\zeta). \end{aligned} \tag{56}$$

This shows that,—

To a group ξ, η *of the polar system of* δ (*viz. to a pair of points satisfying the condition* $(\delta\xi)(\delta\eta) = 0$) *corresponds in our transformation a couple of conjugate forms* $(ax)^4$, $(bx)^4$ *of the pencil* $\varkappa f + \lambda h$. (Such forms as f and ϕ, satisfying the equation $(ab)^4 = 0$.)

To a group ξ, η, ζ *of the polar system of* q *corresponds in our transformation a set of three forms* $(ax)^4$, $(bx)^4$, $(cx)^4$ *of our pencil, bearing the mutual relation*

$$(ab)^2(ac)^2(bc)^2 = 0.$$

We have evidently still a third theorem of this kind, namely:

If the points ξ, η, ζ *form a polar-group of* p, *the biquadratic covariants* ϕ, ψ, χ *of the corresponding forms* $(ax)^4$, $(bx)^4$, $(cx)^4$ *are connected by the equation*

$$(\phi\psi)^2(\phi\chi)^2(\psi\chi)^2 = 0,$$

(and vice versa. Of course ψ and χ mean, with respect to $(bx)^4$ and $(cx)^4$ what ϕ means with respect to $(ax)^4$.) Supposing that ξ, η, ζ coincide with one another, and paying attention to the identity $-27g_3(\phi) = 32G^2$, we fall back on our former results.

Finally we mention the theorem:

The point η, *corresponding to the Hessian* h *of* f, *is the linear polar of the point* ξ, *corresponding to* f *itself, with respect to the cubic* p.

Namely, we obtain the equation

$$(\lambda'\xi)^2.l^2 + (\mu'\xi)^2.m^2 + (\nu'\xi)^2.n^2 = 0,$$

which determines the vanishing points of h, by eliminating η from the equations

$$(p\xi)^2(p\eta) = \frac{r}{3}\cdot\{(\lambda\xi)^2(\lambda\eta) + (\mu\xi)^2(\mu\eta) + (\nu\xi)^2(\nu\eta)\} = 0$$

and

$$(\lambda'\eta).l^2 + (\mu'\eta).m^2 + (\nu'\eta).n^2 = 0.$$

The equation (55) determines the binary quartic corresponding to a given point ξ. The inverse problem, to find the point ξ when the quartic is known, is solved in a similar way by means of the equation

$$0 = (\lambda\xi).l^2 + (\mu\xi).m^2 + (\nu\xi).n^2 \cong -\frac{3}{2\sqrt{G}}\cdot\phi, \tag{55b}$$

which is, in a certain sense, reciprocal to the first one. In order to find the point ξ corresponding to f we have simply to express that the quartic on the right is conjugate to f.

The results derived here from the consideration of the form (55) can also be obtained otherwise without difficulty. But the method is interesting in itself. We have here a very simple, say degenerate, example of an unlimited number of principles of transference, the study of which leads to results of some importance. The central point in these theories is always a form containing variables of two different domains, the variables of which are submitted to independent linear transformations. In our special case it does not seem necessary to establish the complete system of invariants and covariants of the form (55); we may signalize, however, the essential importance of the question.

§9. *Connection between Cubic and Octahedron.*

The one-to-four correspondence between the binary domain (ξ) of the cubic p and the binary domain (x) of the quartic f, the object of the research contained in §8, can be looked at from still a different standpoint, opening a new view of the theory of the octahedron.

We arrive at it by establishing the following theorem:

The one-to-four correspondence between ξ and x defined by the equation

$$(\lambda'\xi).l^3 + (\mu'\xi).m^3 + (\nu'\xi).n^3 = 0 \tag{55}$$

is identical with the one determined by means of the proportion

$$(\lambda\xi):(\mu\xi):(\nu\xi) = l^3:m^3:n^3. \tag{57}$$

Namely, comparing the formula (8), p. 172, and (22), p. 180, and paying regard to the circumstance that when the ratios of l^3, m^3, n^3 are known, four different sets of values are still left for the ratios of l, m, n, we see that (57) actually defines a one-to-four correspondence between ξ and x, the point x being perfectly determined by the ratios of l, m, n.

Writing now in (57) for a moment η instead of ξ, and eliminating l^3, m^2, n^2 from (55) and (57), we obtain

$$0 = (\lambda'\xi)(\lambda\eta) + (\mu'\xi)(\mu\eta) + (\nu'\xi)(\nu\eta) = -3\,(\xi\eta), \tag{58}$$

that is to say, the point η coincides with ξ, and (55) and (57) represent the same transformation.

It follows from this that we are able to put ($\lambda\xi$), etc., equal to $\rho.l^3$, etc., ρ denoting a factor of proportionality. This factor we assume, for certain reasons of homogeneity and partly of convenience, equal to $3\,\frac{\sqrt{-R}}{\sqrt{-r}}$. Substituting now these values of ($\lambda\xi$), etc., in the formulæ of the theory of the cubic, we obtain immediately the following theorem:

The system of rational and irrational covariants of the cubic p is transformed into the system of rational and irrational covariants of the sextic T by means of the *following substitutions, which are consistent with each other, and represent a one-to-four correspondence between the binary domains of* (ξ) *and* (x):

$$\begin{aligned} &\sqrt{-r}\,.\,p \cong \sqrt{-R}\,.\,T^2 \quad , \\ &\delta \cong \Phi \quad , \qquad q \cong \Psi \quad , \end{aligned} \tag{59}$$

$$\begin{aligned} &(\sigma\xi) \cong \mathfrak{S} \quad , \qquad (\tau\xi) \cong \mathfrak{T} \quad , \\ &\sqrt{-r}\,.\,(\sigma'\xi) \cong \sqrt{-R}\,.\,\mathfrak{S}', \quad \sqrt{-r}\,(\tau'\xi) \cong \sqrt{-R}\,.\,\mathfrak{T}', \end{aligned} \tag{60}$$

$$\begin{aligned} &\sqrt{-r}\,.\,(\lambda\xi) \cong 3\sqrt{-R}\,.\,l^2, \qquad (\lambda'\xi) \cong L \quad , \\ &\sqrt{-r}\,.\,(\mu\xi) \cong 3\sqrt{-R}\,.\,m^2, \qquad (\mu'\xi) \cong M \quad , \\ &\sqrt{-r}\,.\,(\nu\xi) \cong 3\sqrt{-R}\,.\,n^2, \qquad (\nu'\xi) \cong N \quad . \end{aligned} \tag{61)*$$

This beautiful result is not surprising. The theory of the irrational covariants of the cubic is based upon the syzygy

$$rp^2 + 4\delta^3 + q^2 = 0 \qquad \text{(Nr. 2, p. 171)},$$

whereas the theory of the covariants of the octahedron is derived from the syzygy

$$RT^4 + 4\Phi^3 + \Psi^2 = 0 \qquad \text{(Nr. 10, p. 178)}.$$

From the close resemblance between these formulæ follows, that in both theories we had to perform, step by step, the same operations; and so we cannot wonder at the fact that finally the results can be transformed into each other by a simple principle of transference. We have not thought it convenient, however, to choose this remark as our starting point, for by doing so we should have lost the connection with the results developed in §8, which is an essential feature of our theory.

Our principle of transference appears in two different forms, a rational (59) and an irrational one (60) and (61). In (60) and (61) we may exchange l^2, m^2 and n^2 *ad libitum*; so we see that the formulæ (59), without the supplementary formulæ (60) and (61), represent a transformation 6 to 24, which is decomposed into six different transformations by adjunction of the irrational covariants l^2, m^2, n^2.

Finally, we mention that the involutory linear transformation, which exchanges the vanishing points of p with the corresponding vanishing points

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[*] The aspect of these formulæ is somewhat simplified by the supposition $\sqrt{-r} \cong \sqrt{-R}$, which is legitimate, since no relation between r and R follows from (59)-(61). But thus we should only disguise the irrationality, and destroy the homogeneity of our formulæ with respect to symbols of p and T.

of q, has its equivalent in an involutory four-to-four transformation of the binary domain (x) which transforms the vanishing points of T into those of Ψ, and exchanges the irrational covariants of T among each other.

§10. *The Linear Factors $(r_\kappa x)$ of the Quartic.*

It is easy to determine a set of four linear forms whose vanishing points are identical with those of the quartic f. Namely the squares of the said forms can be expressed linearly in terms of the quadratic forms l, m, n, which are linearly independent; their discriminants vanish; finally, the fourth powers of the linear forms in question are conjugate to f. By these properties the squares of the said forms are defined, save factors not depending on $x(x_1:x_2)$.

We find it convenient to choose these factors in two different manners, adapted to different purposes, and accordingly to introduce two different sets of notations $(\rho_\kappa x)$, $(r_\kappa x)$ for what we may term "linear factors of f."

Thus we write

$$\begin{aligned}
\sqrt[4]{G}.(\rho_0 x)^2 &= -2(r_0 x)^2 = \sqrt{e_\mu - e_\nu}.l + \sqrt{e_\nu - e_\lambda}.m + \sqrt{e_\lambda - e_\mu}.n, \\
\sqrt[4]{G}.(\rho_\lambda x)^2 &= -2(r_\lambda x)^2 = \sqrt{e_\mu - e_\nu}.l - \sqrt{e_\nu - e_\lambda}.m - \sqrt{e_\lambda - e_\mu}.n, \\
\sqrt[4]{G}.(\rho_\mu x)^2 &= -2(r_\mu x)^2 = -\sqrt{e_\mu - e_\nu}.l + \sqrt{e_\nu - e_\lambda}.m - \sqrt{e_\lambda - e_\mu}.n, \\
\sqrt[4]{G}.(\rho_\nu x)^2 &= -2(r_\nu x)^2 = -\sqrt{e_\mu - e_\nu}.l - \sqrt{e_\nu - e_\lambda}.m + \sqrt{e_\lambda - e_\mu}.n.
\end{aligned} \tag{62}$$

Here the square roots of the quantities $e_\mu - e_\nu$, etc., may be chosen arbitrarily, whereas $\sqrt[4]{G}$ means the product of the said square roots:

$$\sqrt[4]{G} = \sqrt{e_\mu - e_\nu}.\sqrt{e_\nu - e_\lambda}.\sqrt{e_\lambda - e_\mu}. \tag{63}$$

The quantities $(r_\kappa x)$, as defined herewith, are evidently roots of an equation of the degree 32 in the original domain of rationality of f, whereas the quantities $(\rho_\kappa x)$ satisfy an equation of the 8th degree only, which is the lowest degree possible for linear irrational covariants proportional to the linear factors of f.

In order to establish this equation, we investigate the values of the symmetric functions

$$\begin{aligned}
\sum (r_\iota x)^2 &= 0 \quad , \\
\sum (r_\iota x)^2 (r_\kappa x)^2 &= \tfrac{3}{4} \frac{\Phi}{\sqrt{G}}, \\
\sum (r_\iota x)^2 (r_\kappa x)^2 (r_l x)^2 &= \tfrac{1}{2} \frac{t}{\sqrt{G}}, \\
(r_0 x)(r_\lambda x)(r_\mu x)(r_\nu x) &= \tfrac{1}{4}.f \quad .
\end{aligned} \tag{64}$$

Of course the last of these equations is, strictly speaking, not a consequence of the equations (62), which furnish only the value of the product $\Pi(r_\kappa x)^2$. But being able to extract the square root, we may, as we have actually done, choose one of the two values of $\Pi(r_\kappa x)$ at random.

From (64) follows:

The squares of the linear forms $(\rho_\kappa x)$ *are the roots of the biquadratic equation*

$$G.y^4 + * + 6\phi.y^2 + 4t.y + f^2 = 0, \tag{65}$$

which defines the forms $(\rho_\kappa x)$ and $(r_\kappa x)$ as irrational covariants of the degrees $-\frac{1}{2}$ and $\frac{1}{4}$ respectively.

The solution of this equation depends, as we have seen, upon the solution of the cubic equation No. 16, which is free from the parameter $(x_1:x_2)$, and upon two subsequent operations which are independent of each other: First, upon the extraction of the square root of two of the quantities l^2, m^2, n^2 (see No. 31), which contain the parameter, but are combinants of the pencil $\varkappa f + \lambda h$; secondly, upon the extraction of the three square roots $\sqrt{e_\mu - e_\nu}$, etc., which are again free from the parameter.*

The formulæ (62)–(65) contain not only the decomposition of $\frac{1}{4}f$ or $\frac{f}{\sqrt{G}}$ into linear factors by means of invariant processes, but they solve also the corresponding problem for *all the forms of the pencil* $\varkappa f + \lambda h$: l, m, n being combinants, we have in our formulæ simply to replace G by $G_{\varkappa\lambda}$ and $e_\mu - e_\nu$ etc. by $(e_\mu - e_\nu)(\varkappa - \lambda e_\lambda)$ etc. In special cases these expressions may of course be simplified; for instance, when the linear factors of T, Φ, Ψ are in question.

The decomposition of ϕ into linear factors depends upon the solution of the equation

$$g_3^2.y^4 - 4G.g_3.f.y^2 + 2G.g_3.t.y + \frac{G}{4}.\phi^2 = 0; \tag{65b}$$

the linear factors are defined by

$$\sqrt{g_3}.(\rho_0'x)^2 = \sqrt{G}.\{\sqrt{e_\lambda}.l + \sqrt{e_\mu}.m + \sqrt{e_\nu}.n\}, \text{ etc.} \tag{62b}$$

We are further enabled to decompose the forms of the 24th degree contained in the pencil (RT^4, Φ^3, Ψ^2) which are most conveniently written in the special shape

$$4\Psi^2 - g_3^2.T^4 = -\tfrac{1}{27}g_2^3.T^4 - 16\Phi^3$$

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[*] It would be desirable to derive this result from the direct solution of (65) in terms of radicals.

into their linear factors. We have first to split up such a form into its six conjugate biquadratic factors; the solution of this problem is derived from the formulæ on p. 193 by means of our principle of transference; secondly, we have to apply the formulæ (62)–(65).*

§11. *The Irrational Invariants* $(r_\iota r_\kappa)$.

Let us now operate in the domain of rationality defined by the quantities

$$\sqrt{e_\mu - e_\nu},\ \sqrt{e_\nu - e_\lambda},\ \sqrt{e_\lambda - e_\mu};\ l,\ m,\ n.$$

Here the covariants $(r_\kappa x)^2$ are fully determined; besides we know the value of the product $(r_0 x)(r_\lambda x)(r_\mu x)(r_\nu x)$. Hence we have altogether *eight* different sets of values of the linear covariants $(r_\kappa x)$.

Passing, by proper changes of sign, from one of these sets to the others, we notice that the simultaneous invariants $(r_\iota r_\kappa)$ assume only four different sets of values. This leads to an important remark: *The simultaneous invariants* $(r_\iota r_\kappa)$ *belong to the domain of the quantities* $\sqrt{e_\mu - e_\nu}$, $\sqrt{e_\nu - e_\lambda}$, $\sqrt{e_\lambda - e_\mu}$.

Indeed, let us calculate the simultaneous invariants and covariants of the quadratic forms $(r_\kappa x)^2$, defined by (62).

First, the Jacobians of any two of these forms will be found by means of the formulæ (20):

$$-2(r_0 r_\lambda)(r_0 x)(r_\lambda x) = \sqrt{e_\mu - e_\nu}.\{\sqrt{e_\lambda - e_\mu}.m - \sqrt{e_\nu - e_\lambda}.n\},$$
$$-2(r_\mu r_\nu)(r_\mu x)(r_\nu x) = \sqrt{e_\mu - e_\nu}.\{\sqrt{e_\lambda - e_\mu}.m + \sqrt{e_\nu - e_\lambda}.n\}.$$

Comparing the product of these two expressions with the equation

$$4(r_0 x)(r_\lambda x)(r_\mu x)(r_\nu x) = f \qquad \text{(No. 64)},$$

we have

$$(r_0 r_\lambda).(r_\mu r_\nu) = -(e_\mu - e_\nu), \text{ etc.} \tag{66a}$$

Moreover we find, by means of (21),

$$(r_0 r_\lambda)^2 = (r_\mu r_\nu)^2 = (e_\mu - e_\nu), \text{ etc.,} \tag{66b}$$

$$\left.\begin{array}{ll} (r_\mu r_\nu)(r_\nu r_\lambda)(r_\lambda r_\mu) = \sqrt[4]{G}, & (r_\lambda r_0)(r_0 r_\mu)(r_\mu r_\lambda) = \sqrt[4]{G}, \\ (r_\nu r_\lambda)(r_\lambda r_0)(r_0 r_\nu) = -\sqrt[4]{G}, & (r_0 r_\mu)(r_\mu r_\nu)(r_\nu r_0) = -\sqrt[4]{G}. \end{array}\right. \tag{66c}$$

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* Compare Klein's Icosahedron. The standpoint of this work is, however, not quite identical with ours. Klein operates throughout with special systems of coordinates, whereas in our considerations the coordinates remain perfectly general.

By these equations actually four systems of values are left to the simultaneous invariants $(r_\iota r_\kappa)$, which differ from one another by changes of sign, and correspond exactly to the available changes of sign in the forms $(r_\iota x)$.

Now we may choose one of the four sets of values at random, assuming for instance

$$\begin{aligned}(r_\mu r_\nu) &= -(r_0 r_\lambda) = \sqrt{e_\mu - e_\nu},\\ (r_\nu r_\lambda) &= -(r_0 r_\mu) = \sqrt{e_\nu - e_\lambda},\\ (r_\lambda r_\mu) &= -(r_0 r_\nu) = \sqrt{e_\lambda - e_\mu}.\end{aligned} \tag{67}$$

Hereby, of course, also the ambiguity of the quantities $(r_\kappa x)$ is lessened, only a simultaneous change of sign being left to the whole set.

Thus in the domain of the quantities l, m, n, $\sqrt{e_\mu - e_\nu}$, $\sqrt{e_\nu - e_\lambda}$, $\sqrt{e_\lambda - e_\mu}$, *the linear forms* $(r_\kappa x)$ *can be defined as a two-valued set of four quantities.*

This theorem, which seems to have been overlooked hitherto, becomes important when the connection of the theory of the quartic with the theory of *elliptic functions* is in question. The simplicity of the formulæ by means of which we shall express this connection in a subsequent paper, is partly due to the circumstance that we are able to replace the comparatively complicated formulæ (66) by the simpler formulæ (67). Here, as well as in the case of the last equation (64), where we had a first choice among two possibilities, we have made our assumptions so as to make the formulæ expressing the said connection as simple as possible.

From the formulæ (67) we derive the well-known expressions for the *double-ratios* (anharmonic ratios) *of the four points* r_0, r_λ, r_μ, r_ν:

$$\frac{(r_0 r_\nu) \cdot (r_\lambda r_\mu)}{(r_\lambda r_\nu) \cdot (r_0 r_\mu)} = \frac{e_\mu - e_\lambda}{e_\nu - e_\lambda}, \text{ etc.} \tag{68}$$

The quantities e_λ, e_μ, e_ν are already known at the same time with the products $\psi_\lambda = c \,.\, (r_0 x)(r_\lambda x)$, $\chi_\lambda = \frac{1}{c} \,.\, (r_\mu x)(r_\nu x)$, etc., c denoting an arbitrary parameter:

$$2\,(\psi_\lambda, \chi_\lambda)_2 = -3e_\lambda.^* \tag{69}$$

§12. *Further Properties of the Linear Forms* $(r_\kappa x)$.

There is a linear identity among any three binary linear forms, the coefficients of which are the simultaneous invariants of these forms. We may write down the peculiar shape the identities among the forms $(r_\kappa x)$ assume on account of the relations (67):

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*Compare F. Klein, Math. Annalen, v. 27, p. 459, No. 64. The numerical coefficient $\frac{1}{6}$ in Klein's formula is erroneous. The true value is $-\frac{1}{6}$.

$$\begin{aligned} (r_\mu r_\nu).(r_\lambda x) + (r_\nu r_\lambda).(r_\mu x) + (r_\lambda r_\mu).(r_\nu x) &= 0, \\ (r_\mu r_\nu).(r_0 x) + (r_\lambda r_\mu).(r_\mu x) - (r_\nu r_\lambda).(r_\nu x) &= 0, \\ -(r_\lambda r_\mu).(r_\lambda x) + (r_\nu r_\lambda).(r_0 x) + (r_\mu r_\nu).(r_0 x) &= 0, \\ (r_\nu r_\lambda).(r_\lambda x) - (r_\mu r_\nu).(r_\mu x) + (r_\lambda r_\mu).(r_0 x) &= 0. \end{aligned} \tag{70}$$

These formulæ are special cases of the following set of four identities, containing two sets of variables $x\,(x_1 : x_2)$ and $y\,(y_1 : y_2)$:

$$\begin{aligned} (r_0 x)(r_0 y) + (r_\lambda x)(r_\lambda y) + (r_\mu x)(r_\mu y) + (r_\nu x)(r_\nu y) &= 0, \\ (r_0 x)(r_\lambda y) - (r_\lambda x)(r_0 y) + (r_\mu x)(r_\nu y) - (r_\nu x)(r_\mu y) &= 0, \\ (r_0 x)(r_\mu y) - (r_\mu x)(r_\nu y) + (r_\nu x)(r_\lambda y) - (r_\lambda x)(r_\nu y) &= 0, \\ (r_0 x)(r_\nu y) - (r_\nu x)(r_0 y) + (r_\lambda x)(r_\mu y) - (r_\mu x)(r_\lambda y) &= 0. \end{aligned} \tag{71}$$

These remarkable equations are algebraic consequences of the equations (70); the first among them can also be derived from the first equation in (64), while the others immediately follow from the equations (67). (If, in the last formula (64), we should prefer to write $-f$ instead of f, or if we should replace the solution (67) of the equations (66) by another of the four possible solutions, the equations (70) and (71) would, of course, be changed into a different but similar set of equations.)

Worth noticing are further the expressions of the polars $(lx)(ly)$, etc., in terms of the linear forms $(r_\kappa x)(r_\kappa y)$:

$$(lx)(ly) \begin{cases} = -\dfrac{(r_0 x)(r_0 y) + (r_\lambda x)(r_\lambda y)}{(r_\mu r_\nu)} = \dfrac{(r_\mu x)(r_\mu y) + (r_\nu x)(r_\nu y)}{(r_\mu r_\nu)} \\ = \dfrac{(r_0 x)(r_\mu y) + (r_\nu x)(r_\lambda y)}{(r_\mu r_\lambda)} = \dfrac{(r_\mu x)(r_0 y) + (r_\lambda x)(r_\nu y)}{(r_\mu r_\lambda)} \\ = \dfrac{(r_0 x)(r_\nu y) - (r_\mu x)(r_\lambda y)}{(r_\nu r_\lambda)} = \dfrac{(r_\nu x)(r_0 y) - (r_\lambda x)(r_\mu y)}{(r_\nu r_\lambda)}. \end{cases} \tag{72}$$

To these formulæ we may add the following expressions of the products $(mx)(my).(nx)(ny)$ (which we have already recognized as polars (p. 181)), together with the expressions of the products $(xy).(lx)(ly)$, etc.:

$$\begin{aligned} (r_\nu r_\lambda)(r_\lambda r_\mu).(mx)(my).(nx)(ny) &= (r_\mu x)(r_\nu x).(r_\mu y)(r_\nu y) - (r_0 x)(r_\lambda x).(r_0 y)(r_\lambda y), \\ (r_\nu r_\lambda)(r_\lambda r_\mu).(xy).(lx)(ly) &= (r_0 x)(r_\lambda x).(r_\mu y)(r_\nu y) - (r_\mu x)(r_\nu x).(r_0 y)(r_\lambda y), \end{aligned} \tag{73}$$

The formulæ (72) put the well-known fact into evidence that the vanishing points of the form $(lx)^2$ are harmonically separated by the vanishing points r_0, r_λ and r_μ, r_ν of f. The same thing is expressed by the formulæ

$$(lr_0)(lr_\lambda)=0\,,\;\;(lr_\mu)(lr_\nu)=0\,, \tag{74}$$

or by the formulæ

$$\begin{aligned}(lr_0)(lx)&=-(r_\lambda x)\,, & (lr_\mu)(lx)&=-(r_\nu x)\,,\\ (lr_\lambda)(lx)&=(r_0 x)\,, & (lr_\nu)(lx)&=(r_\mu x)\,,\end{aligned} \tag{75}$$

$$4.\big((r_0x)(r_\lambda x)\,,\;(r_\mu x)(r_\nu x)\big)_1=(r_\nu r_\lambda).(r_\lambda r_\mu).(lx)^2. \tag{76}$$

Finally, among the expressions of covariants of f in terms of the forms $(r_\kappa x)$, the following expression of the covariant ϕ is worth noticing:

$$-\tfrac{3}{2}\frac{\Phi}{\sqrt{G}}=(r_0x)^4+(r_\lambda x)^4+(r_\mu x)^4+(r_\nu x)^4. \tag{77}$$

It shows immediately that ϕ is conjugate to f.

In all these developments, the linear forms $(r_\kappa x)$ are considered as irrational covariants of a given quartic f. But we may just as well choose a set of four given linear forms as starting point. Adapting these forms to the relations (67) we obtain the same theorems, as stated before, arranged in the opposite order.

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On the Connection between Binary Quartics and Elliptic Functions.

By E. Study.

The object of the following research is an application of the theory developed in the preceding paper to elliptic functions.

The binary quartic has played an important part in elliptic functions from the very beginnings of this theory, and the subject was early considered from the standpoint of the modern theory of invariants. A prominent result was Hermite's famous transformation;[*] a pair of important formulæ has been communicated by Weierstrass in his lectures;[†] some no less important results were added by F. Klein,[‡] who at the same time applied similar methods to hyperelliptic functions, and originated in this way a series of investigations of quite a new character; finally, we have to mention a dissertation of Burkhardt,[§] who extended his considerations to some irrational covariants.

Our starting point is different from those of the authors mentioned, and in some respects more elementary. We compare the relations among the rational and irrational covariants of a quartic with the identities among the four Θ-functions. Simple as this idea may be, nevertheless a new light is thrown by it upon the familiar formulæ, and at the same time a number of new results can be derived, which make the theory in question in a certain sense *complete.*

The method applied being so elementary, we need not dwell upon the details of the proofs; but we may lay some stress upon the fact that all our results are obtained by means of *actual calculations*, referring to the *general* case of a binary quartic (not merely to a canonical form), and that no use whatever is made of what is called the method of indeterminate coefficients. Neither do we take results from one theory and apply them to the other. Such proceedings are excellent means of investigation when new fields are opened, but in a more advanced state of science they are hardly satisfactory. The interest of the connection between quartics and elliptic functions is, besides, not lessened by the fact that either theory permits the deduction of its theorems by means of its own.

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[*] Crelle's Journal, v. 52. See also Clebsch, Binäre Formen, §62, and Weber, Elliptische Functionen, I, §4.

[†] Biermann, Problema mechanica (Diss., Berlin, 1865).

[‡] Hyperelliptische Sigmafunctionen, Math. Ann., vol. 27, §12, p. 454.

[§] Beziehungen zwischen der Invariantentheorie und der Theorie der algebraischen Integrale und ihrer Umkehrungen, München, 1887.

The Θ-functions used in this research are not quite identical with Weierstrass' Θ-functions. We refer to the author's paper, "On the Addition Theorems of Jacobi and Weierstrass" (*Am. Journ.*, Vol. XVI, p. 156), the results of which are supposed to be known to the reader.*

The preceding paper is simply quoted as "Quartic."

§1. *The Four Θ-Functions and the Linear Forms* $(r_\kappa x)$.

In order to compare the theory of binary quartics with the theory of elliptic functions, we identify the irrational invariants $\sqrt{e_\mu - e_\nu}$, etc., with the quantities $\sqrt{e_\mu - e_\nu}$, etc., belonging to the latter theory. Then of course our rational invariants g_2, g_3, G coincide with the quantities g_2, g_3, G of Weierstrass. Now comparing the linear relations among the forms $(r_\kappa x)$ (Quartic No. 70) with the identities among the squares of the functions $\Theta_\kappa u$, we see that we are enabled to explain a set of square roots by the formulæ

$$\sqrt{(r_\mu r_\nu)} \cong \Theta_\lambda(0) = \Theta_\lambda, \text{ etc.,} \tag{1}$$

$$\sqrt{\sqrt[4]{G}} = \sqrt[8]{G} \cong \Theta'(0) = \Theta' \tag{2}$$

(where $\Theta' = \Theta_\lambda(0)\,\Theta_\mu(0)\,\Theta_\nu(0) = \Theta_\lambda\Theta_\mu\Theta_\nu$), and

$$\begin{aligned} \sqrt{(r_0 x)} &\cong \Theta u\,, & \sqrt{(r_\lambda x)} &\cong \Theta_\lambda u, \\ \sqrt{(r_\mu x)} &\cong \Theta_\mu u, & \sqrt{(r_\nu x)} &\cong \Theta_\nu u. \end{aligned} \tag{3}$$

Considering u as a variable,† we have established a one-to-two correspondence between the values of u, situated within the parallelogram of periods, and the points x of the binary domain. When u varies throughout the parallelogram, the point x varies through the binary domain, obtaining each situation twice; *vice versa,* to every point belong two values $\pm u$. Points of exception are only the points $x = r_0, r_\lambda, r_\mu, r_\nu$, to which correspond only single values of u, namely, half periods. And if u varies throughout the plane representing the complex values of u, the forms $(r_\kappa x)$ defining the corresponding point of the binary domain are reproduced, excepting an exponential factor, common to all of them. Proceeding from u to $u' = u \pm 2\omega_\lambda$, $(r_\kappa x)$ is changed into $(r_\kappa x') = e^{4\eta_\lambda\omega_\lambda} . e^{\pm 4\eta_\lambda u} . (r_\kappa x)$. In the same way the square roots $\sqrt{(r_\kappa x)}$ are reproduced with an exponential factor common to all of them, when we surpass the boundaries of the *double* parallelogram of periods. Passing from u to $u' = u \pm 4\omega_\lambda$, $\sqrt{(r_\kappa x)}$ is changed into $\sqrt{(r_\kappa x')} = e^{8\eta_\lambda\omega_\lambda} . e^{\pm 4\eta_\lambda u} . \sqrt{(r_\kappa x)}$, as we could partly have anticipated by considering the two-leaved Riemann surface defined by the branchpoints $r_0, r_\lambda, r_\mu, r_\nu$.

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* I use this opportunity to correct a misprint: On p. 156 write $\Re\left(\frac{\omega_3}{\omega_1 i}\right)$ and $\Re\left(\frac{\omega_2}{\omega_1 i}\right)$ instead of $\Re\left(\frac{\omega_2}{\omega_1}\right)$ and $\Re\left(\frac{\omega_3}{\omega_1}\right)$.

† It must not be overlooked that after the establishment of equations of the form (3), the binary variables x_1 and x_2, belonging to a definite system of coordinates, are no longer independent quantities, although the ratio $x_1 : x_2$ continues to be arbitrary. Namely, when the coefficients of the forms $(r_\kappa x)$ are

Now we may operate principally upon the surface just mentioned, introducing the new irrationality $\sqrt{f}$ by the formula

$$\sqrt{f} = -2\sqrt{(r_0 x)}\sqrt{(r_\lambda x)}\sqrt{(r_\mu x)}\sqrt{(r_\nu x)}. \tag{4}$$

Considering f and its covariants as functions of u, we draw immediately the conclusions:

1). *f and all rational covariants of f, and certain irrational covariants besides, are one-valued functions of the argument u, which are reproduced with an exponential factor, when u is augmented by a period.*

2). *All quotients of such covariants, the degree of which in x is zero (viz. all absolute covariants of f), are elliptic functions of u.*

Of course the formulæ (3), the basis of our further considerations, cannot claim any novelty, as far as their general form is concerned.* We may call attention, however, to the circumstance that we have made special suppositions in the definition of the linear forms $(r_\kappa x)$ on the one side and in the definition of

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known, the periods 2ω and $2\omega'$ can be determined as functions of these coefficients. Substituting, then, a definite value of u in the functions $\Theta_\kappa u$, not only the ratio of x_1 and x_2 is determined, but also the absolute values of these quantities. Starting therefore from *given* values of x_1 and x_2, we have to add a factor of proportionality, say $\rho(x)$. It does not seem necessary, however, to carry this factor visibly through the whole investigation; we may unite it with the variables x_1 and x_2 and denote the products ρx_1 and ρx_2 again by x_1 and x_2.—See the footnotes in §2 and §6.

* Compare Weber, Elliptische Functionen und algebraische Zahlen (Braunschweig, 1893), §86.

our Θ-functions on the other. It is entirely due to the exact parallelism in the two definitions that we are enabled to derive from these formulæ *simple* consequences; an apparently slight change of notation would imply considerable algebraical complications.

§2. *The Elliptic Functions* $\wp u$, $\wp' u$.

We now proceed to establish the expressions of the elliptic functions $\wp u$, $\wp' u$ in terms of covariants of f. Considering the distribution of the argument u on the Riemann surface defined by $\sqrt{f}$, we cannot expect to find the values of these functions in the domain of $\sqrt{f}$ and the rational covariants of f; whereas, as we shall see, the contrary holds in the case of the functions of the double argument $\wp(2u)$, $\wp'(2u)$.

Expressing $\wp(u)$, etc., in terms of the Θ-functions, we find, after a short calculation,

$$\begin{aligned}
&\frac{(r_\nu r_\lambda)(r_\lambda r_\mu).(r_\lambda x) + (r_\lambda r_\mu)(r_\mu r_\nu).(r_\mu x) + (r_\mu r_\nu)(r_\nu r_\lambda).(r_\nu x)}{3\,(r_0 x)} = \\
&\qquad = \frac{(tr_0)^5\,(tx)}{2\sqrt{G}.(r_0 x)} = \frac{(ar_0)^2\,(ax)^2}{2\,(r_0 x)^2} \cong \wp u, \\
&\frac{-(r_\nu r_\lambda)(r_\lambda r_\mu).(r_0 x) + (r_\lambda r_\mu)(r_\mu r_\nu).(r_\nu x) - (r_\mu r_\nu)(r_\nu r_\lambda).(r_\mu x)}{3\,(r_\lambda x)} = \\
&\qquad = \frac{(tr_\lambda)^5\,(tx)}{2\sqrt{G}.(r_\lambda x)} = \frac{(ar_\lambda)^2\,(ax)^2}{2\,(r_\lambda x)^2} \cong \wp\,(u \pm \omega_\lambda),
\end{aligned} \tag{5}$$

$$\begin{aligned}
&-2\,(r_\mu r_\nu)(r_\nu r_\lambda)(r_\lambda r_\mu).\frac{\sqrt{(r_\lambda x)}\sqrt{(r_\mu x)}\sqrt{(r_\nu x)}}{\sqrt{(r_0 x)}\sqrt{(r_0 x)}\sqrt{(r_0 x)}} = \frac{\sqrt{2\,(tr_0)^3\,(tx)^3}}{\sqrt{(r_0 x)}\sqrt{(r_0 x)}\sqrt{(r_0 x)}} = \\
&\qquad = \sqrt[4]{G}.\frac{\sqrt{f}}{(r_0 x)^2} = \frac{(ar_0)^3(ax).\sqrt{f}}{(r_0 x)^3} \cong \wp' u, \\
&-2\,(r_\mu r_\nu)(r_\nu r_\lambda)(r_\lambda r_\mu).\frac{\sqrt{(r_0 x)}\sqrt{(r_\nu x)}\sqrt{(r_\mu x)}}{\sqrt{(r_\lambda x)}\sqrt{(r_\lambda x)}\sqrt{(r_\lambda x)}} = \frac{\sqrt{2\,(tr_\lambda)^3\,(tx)^3}}{\sqrt{(r_\lambda x)}\sqrt{(r_\lambda x)}\sqrt{(r_\lambda x)}} = \\
&\qquad = \sqrt[4]{G}.\frac{\sqrt{f}}{(r_\lambda x)^2} = \frac{(ar_\lambda)^3(ax).\sqrt{f}}{(r_\lambda x)^3} \cong \wp'\,(u \pm \omega_\lambda),
\end{aligned} \tag{6}$$

$$\begin{aligned}
&\frac{3\,(tr_0)^4(tx)^2}{\sqrt[4]{G}.(r_0 x)^2} = 2\sqrt[4]{G}.\frac{(ar_0)(ax)^3}{(r_0 x)^3} \cong \wp'' u, \\
&\frac{3\,(tr_\lambda)^4\,(tx)^2}{\sqrt[4]{G}.(r_\lambda x)^2} = 2\sqrt[4]{G}.\frac{(ar_\lambda)(ax)^3}{(r_\lambda x)^3} \cong \wp''\,(u \pm \omega_\lambda),
\end{aligned} \tag{7}$$

etc.

To facilitate the understanding of the significance of these formulæ we call to mind the following relations among the elliptic functions on the right side of (5), (6) and (7):

$$\wp\,(u \pm \omega_\lambda) = \frac{(e_\mu - e_\lambda)(e_\nu - e_\lambda)}{\wp u - e_\lambda} + e_\lambda,$$

$$\wp'\,(u \pm \omega_\lambda) = -\,(e_\mu - e_\lambda)(e_\nu - e_\lambda) . \frac{\wp' u}{(\wp u - e_\lambda)^2},$$

$$\wp' u . \wp'\,(u \pm \omega_\lambda) . \wp'\,(u \pm \omega_\mu) . \wp'\,(u \pm \omega_\nu) = 16\,G,$$

$$\frac{1}{\wp' u} + \frac{1}{\wp'\,(u \pm \omega_\lambda)} + \frac{1}{\wp'\,(u \pm \omega_\mu)} + \frac{1}{\wp'\,(u \pm \omega_\nu)} = 0,$$

$$\wp'' u = 6\wp^2 u - \tfrac{1}{2}\,g_2.$$

§3. *The Θ-Functions of the Double Argument.*

Passing to the double argument, we obtain a very much simpler set of formulæ:

$$\sqrt{f} \cong -\Theta' . \Theta\ (2u),^* \tag{8}$$

$$\left.\begin{aligned} l &\cong -\Theta_\lambda . \Theta_\lambda\,(2u), \\ m &\cong -\Theta_\mu . \Theta_\mu\,(2u), \\ n &\cong -\Theta_\nu . \Theta_\nu\,(2u); \end{aligned}\right\} \tag{9}$$

consequently we have

$$t \cong 2\Theta'^5 . \Theta_\lambda\,(2u)\,\Theta_\mu\,(2u)\,\Theta_\nu\,(2u), \tag{10}$$

$$t\sqrt{f} \cong -\,2\Theta'^7 . \Theta\,(4u), \tag{11}$$

and

$$-\frac{h}{f} \cong \wp\,(2u), \tag{12}$$

$$\frac{t}{f\sqrt{f}} \cong \wp'\,(2u), \tag{13}$$

$$-3\,\frac{(f,\,t)}{f^2} = 6\,\frac{h^2}{f^2} - \tfrac{1}{2}\,g_2 \cong \wp''\,(2u), \tag{14}$$

$$12\,\frac{(f,(f,\,t))}{f^2\sqrt{f}} = -\,12\,\frac{h.t}{f^2\sqrt{f}} \cong \wp'''\,(2u), \text{ etc.} \tag{15}$$

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* **From (3) and (8) follows:**

$$\frac{\sqrt{(r_\kappa x)}}{\sqrt[8]{f}} \cong \frac{\Theta_\kappa u}{\sqrt[4]{-\Theta' . \Theta\,(2u)}} \qquad (\kappa = 0, \lambda, \mu, \nu).$$

These formulæ are able to replace the formulæ (3) in most respects. They are independent of the above supposition concerning the values of the binary variables.

The formulæ (12) and (13) are immediately evident; the higher differential quotients of $\wp(2u)$ are easily obtained by means of Gordan's series (Clebsch, Binäre Formen, §8; Gordan's Vorlesungen, v. II, §7), see §6, No. 47.

The corresponding expressions of $\wp(2u+\omega_\lambda)$, etc., contain, of course, the irrationality e_λ:

$$-\frac{f}{l^3}+e_\lambda \cong \wp(2u+\omega_\lambda), \; . \tag{12b}$$

$$-\frac{\sqrt{f}.t}{(e_\mu-e_\lambda)(e_\nu-e_\lambda).l^4}=-2(e_\mu-e_\nu).\frac{\sqrt{f}.mn}{l^3}\cong\wp'(2u+\omega_\lambda). \tag{13b}$$

The formula (12) is due to Hermite; it contains the transformation of the quartic f to Weierstrass' canonical form (Quartic No. 14, 19) by rational operations. It shows that such sets of eight values of u as $\pm u$, $\pm u+\omega_\lambda$, $\pm u+\omega_\mu$, $\pm u+\omega_\nu$ correspond to the vanishing points of the forms $\varkappa f+\lambda h$; *the values of* u, *corresponding to a given form of our pencil, are therefore the roots of the transcendental equation*

$$\wp(2u)=\frac{\varkappa}{\lambda}.$$

Especially to the vanishing points of f itself ($\lambda=0$) correspond, as we have seen already, the demi-periods; whereas the primitive quarters of a period, the roots of the equation

$$(\wp(2u)-e_\lambda)(\wp(2u)-e_\mu)(\wp(2u)-e_\nu)=0$$

correspond to the vanishing points of the sextic t (No. 10, 13). The vanishing points of $\Phi=\frac{1}{2}(t,t)_2$ are the roots of the equation

$$\begin{aligned}0&=\tfrac{1}{3}g_2.\wp'^2(2u).[\wp(2u+\omega_\lambda)+\wp(2u+\omega_\mu)+\wp(2u+\omega_\nu)]=\\&=4g_2[g_2.\wp(2u)^2+3g_3.\wp(2u)+\tfrac{1}{12}g_2^2]=\\&=\left[2g_2.\wp(2u)+3g_3+\tfrac{1}{3}\sqrt{-3}\sqrt{G}\right]\left[2g_2.\wp(2u)+3g_3-\tfrac{1}{3}\sqrt{-3}\sqrt{G}\right]\end{aligned}$$

(Quartic No. 13, 48), and the vanishing points of $\Psi=(t,(t,t)_2)_1$ are the roots of the equation

$$\begin{aligned}0&=2g_3.\wp(2u)^3+\frac{g_2^2}{3}.\wp(2u)^2+\frac{g_2g_3}{2}.\wp(2u)-\left(\frac{g_2^3}{108}-\frac{g_3^2}{2}\right)=\\&=\left[2e_\lambda.\wp(2u)-\frac{g_2}{3}+2e_\lambda^2\right]\left[2e_\mu.\wp(2u)-\frac{g_2}{3}+2e_\mu^2\right]\left[2e_\nu.\wp(2u)-\frac{g_2}{3}+2e_\nu^2\right]\end{aligned}$$

(Quartic No. 37, 43). Finally we may consider the group of eight points defined by the equations $\wp u = 0$, $\wp(u + \omega_\lambda) = 0$, etc. It corresponds to the one form of our pencil, the linear factors of which are proportional to the linear forms appearing in the numerator of the equations (5). This is the quartic

$$g_2^3 f - 16 g_3 . h,$$

as we see by means of the expression of $\wp(2u)$ in terms of $\wp u$, or also by direct calculation, starting from the formulæ (5).

The formulæ (13)–(15), and the expressions of $\wp(2u) - e_\lambda$ to be derived from (8) and (9), have been communicated by Burkhardt in his dissertation. But he fails to give the formulæ (8)–(11); and his formulæ as well as his demonstrations contain a number of, as it seems to me, superfluous complications. Some of his results have been found independently by Harkness and Morley (A Treatise on the Theory of Functions, New York, 1893, §203).

§4. *Formulæ with Two Arguments u, v.*

The results contained in §3 are capable of an important generalization. Writing in No. (3) y and v instead of x and u, and considering the Θ-functions of u and v at the same time, we obtain immediately the following remarkable set of formulæ:

$$\begin{aligned} (xy) &\cong - \Theta\;(u+v)\,\Theta\;(u-v), \\ (lx)(ly) &\cong - \Theta_\lambda(u+v)\,\Theta_\lambda(u-v), \\ (mx)(my) &\cong - \Theta_\mu(u+v)\,\Theta_\mu(u-v), \\ (nx)(ny) &\cong - \Theta_\nu(u+v)\,\Theta_\nu(u-v). \end{aligned} \tag{16}$$

Hence we derive

$$\begin{gathered} (tx)^3(ty)^3 \cong 2\Theta'^4 . \Theta_\lambda(u+v)\,\Theta_\lambda(u-v)\,\Theta_\mu(u+v)\,\Theta_\mu(u-v)\,\Theta_\nu(u+v)\,\Theta_\nu(u-v), \\ 2\,(xy).(tx)^3(ty)^3 \cong - \Theta'^4 . \Theta\,(2u+2v)\,\Theta\,(2u-2v), \end{gathered} \tag{17}$$

and

$$- \frac{\sqrt{(r_0x)}\sqrt{(r_\lambda x)}.\sqrt{(r_\mu y)}\sqrt{(r_\nu y)} \mp \sqrt{(r_0 y)}\sqrt{(r_\lambda y)}.\sqrt{(r_\mu x)}\sqrt{(r_\nu x)}}{(xy)} \cong$$

$$\cong \Theta_\mu \Theta_\nu . \frac{\Theta_\lambda(u \pm v)}{\Theta(u \pm v)} = \frac{\sigma_\lambda(u \pm v)}{\sigma(u \pm v)} = \sqrt{\wp(u \pm v) - e_\lambda}, \tag{18}$$

$$-\frac{(ly)^2\sqrt{(ax)^4}\mp(lx)^2\sqrt{(ay)^4}}{(xy)^2}\cong\frac{\wp'(u\pm v)}{\sqrt{e_\nu-e_\lambda}\sqrt{e_\lambda-e_\mu}}\cdot\frac{\sqrt{\wp(u\mp v)-e_\lambda}}{\sqrt{\wp(u\pm v)-e_\lambda}}, \tag{19}$$

$$\frac{(ax)^2(ay)^2\mp\sqrt{(ax)^4}\sqrt{(ay)^4}}{2(xy)^2}\cong\wp(u\pm v), \tag{20}$$

$$-\frac{(ay)^3(ax)\sqrt{(ax)^4}\mp(ax)^3(ay)\sqrt{(ay)^4}}{(xy)^3}\cong\wp'(u\pm v). \tag{21}$$

Including in the "system of f" the irrationality $\sqrt{f}$, we draw the conclusion:

All covariants of the quartic f containing the two sets of variables x and y in the degree zero, are rational functions of $\wp(u+v)$, $\wp(u-v)$, $\wp'(u+v)$, $\wp'(u-v)$, and vice versa.

Namely, the said covariants are quotients of the integral covariants (xy), $\sqrt{f_x}$, $\sqrt{f_y}$, and of the polars of the forms f, h, t containing x and y. But multiplying such a polar with a properly chosen power of (xy), we obtain an integral function of $(ax)^4$, $(ax)^3(ay)$, $(ax)^2(ay)^2$, $(ax)(ay)^3$, $(ay)^4$. Now considering the formulæ (12), (13), (20), (21), and paying attention to the circumstance that $\wp(2u)$, $\wp'(2u)$, $\wp(2v)$, $\wp'(2v)$ are rational functions of $\wp(u+v)$, etc.,[*] we obtain the above theorem. The inverse statement is immediately evident.

There are, among the absolute covariants in question, a number of quite interesting expressions. We mention the following examples:

$$\frac{(hx)^2(hy)^2}{(xy)^2}\cong-\{\wp(u+v)\wp(u-v)+\tfrac{1}{12}g_2\}, \tag{22}$$

$$\frac{(ax)^3(ay).(hx)(hy)^3-(ay)^3(ax).(hy)(hx)^3}{(xy)^4}=\frac{1}{2}\frac{(ax)^4(hy)^4-(ay)^4(hx)^4}{(xy)^4}=$$
$$=\frac{(tx)^3(ty)^3}{(xy)^3}\cong-\tfrac{1}{2}\wp'(u+v)\wp'(u-v), \tag{23}$$

$$\frac{(hy)^3(hx).\sqrt{(ax)^4}\pm(hx)^3(hy).\sqrt{(ay)^4}}{(xy)^3}\cong\wp(u\pm v)\wp'(u\mp v), \tag{24}$$

$$\frac{(ty)^4(tx)^2.\sqrt{(ax)^4}\pm(tx)^4(ty)^2.\sqrt{(ay)^4}}{(xy)^4}\cong\tfrac{1}{3}\wp'(u\pm v)\wp''(u\mp v), \tag{25}$$

$$\frac{(ax)^3(ay).(hx)(hy)^3+(ay)^3(ax).(hy)(hx)^3}{(xy)^4}=$$
$$=\frac{1}{(xy)^4}.\{(ax)^4.(hy)^4+(ay)^4.(hx)^4-\tfrac{1}{2}g_2.(xy)^2.(ax)^2(ay)^2-\tfrac{2}{3}g_3(xy)^4\} \tag{26}$$
$$\cong-2\{\wp(u+v)\wp(u-v).[\wp(u+v)+\wp(u-v)]+\tfrac{1}{4}g_3\},$$

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[*] See Schwarz, Formeln und Lehrsätze zum Gebrauche der elliptischen Functionen (Göttingen, 1882), Art. 12.

$$\frac{(hx)^4.(hy)^4-g_3.(xy)^2.(ax)^2(ay)^2}{(xy)^4}=\left\{\frac{(hx)^2(hy)^2-{}^1(xy)^2.g_2}{(xy)^2}\right\}^2\cong$$
$$\cong\{\wp(u+v)\wp(u-v)+\tfrac{1}{4}g_2\}^2, \tag{27}$$

$$\frac{(ty)^5(tx).(ax)^4-(tx)^5(ty).(ay)^4}{(xy)^5}=$$
$$=\frac{1}{(xy)}.\left\{\begin{matrix}-8(hx)^4.(hy)^4+\tfrac{2}{3}g_2.(ax)^4.(ay)^4-\\-6g_2.(xy)^2.(hx)^2(hy)^2+9g_3.(xy)^2.(ax)^2(ay)^2\end{matrix}\right\} \tag{28}$$
$$\cong-\tfrac{2}{3}\wp''(u+v)\wp''(u-v)+\tfrac{2}{3}g_2.\wp(u+v)\wp(u-v)$$
$$+g_3\{\wp(u+v)+\wp(u-v)\}+\tfrac{1}{18}g_2^2,$$

$$\frac{(ty)^5(tx).(ax)^4+(tx)^5(ty).(ay)^4}{(xy)^5}=\frac{(ax)^2(ay)^2}{(xy)^2}.2\frac{(tx)^3(ty)^3}{(xy)^3}\cong$$
$$\cong-\{\wp(u+v)+\wp(u-v)\}.\wp'(u+v)\wp'(u-v). \tag{29}$$

Generally, *whenever a power of* (xy) *is the sole denominator, the absolute covariant is an integral function of* $\wp(u+v)$, $\wp(u-v)$, $\wp'(u+v)$, $\wp'(u-v)$. In other respects the properties of a given absolute covariant are determined by the distribution of the radicals $\sqrt{f_x}$, $\sqrt{f_y}$.

The expressions on the left in (20) and (21) were first brought into connection with the elliptic functions $\wp$ and $\wp'$ by Weierstrass;* and his formulæ have been reproduced, with some additional remarks, by F. Klein and others.†

These authors, however, do not seem to have grasped the full content of our formulæ (20) and (21). Namely, instead of considering a pair of arguments $u+v$ and $u-v$, the authors quoted use only one, our $u-v$ (which they denote by $-u$). Thus they consider what we may term the *general distribution of the argument* u upon the Riemann surface defined by $\sqrt{f_x}$ (u having the value zero at an arbitrary point—our point v—of the surface), but *they do not show the relation between this general distribution and the ordinary distribution* (where u vanishes at a branchpoint of the surface), consisting in the coexistence of the formulæ (20), (21) with the formulæ (3), (8), etc. We venture to consider it as an essential improvement that our theory puts the connection between the two distributions into evidence, and thus attributes to all the formulæ a more exact meaning.

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*Biermann, l. c. † F. Klein, Math. Ann., v. 27, §12, 18, p. 454–461.

§5. *Addition Theorems.*

The preceding results permit us, of course, to pursue the parallelism between certain parts of the theory of elliptic functions and the theory of the binary quartic as far as we like. Especially the addition-theorems are easily brought now into a projective form. Take, for instance, three arguments u, v, w, connected by the relation

$$u + v + w \equiv 0 \bmod \tilde{\omega},$$

and denote the corresponding points by x, y, z; then the relations hold

$$\Sigma\,(yz)\,.\,(ty)^3(tz)^3.\sqrt{(ay)^4}\sqrt{(az)^4}.(tx)^6 = 0,$$

and

$$\Omega_{yz} = \Omega_{zx} = \Omega_{xy},$$

where

$$\Omega_{yz} = \frac{(ty)^6.\,(az)^4.\sqrt{(az)^4} - (tz)^6.(ay)^4\sqrt{(ay)^4}}{(yz)\,.\,(ty)^3(tz)^3.\sqrt{(ay)^4}\sqrt{(az)^4}};$$

and when especially

$$u + v + w \equiv 0 \bmod 2\tilde{\omega},$$

we have

$$\Sigma\,(yz)\,.\,(r_0y)(r_0z)\,.\sqrt{(ax)^4} = 0,$$

or

$$\Sigma\,(yz)\,.\sqrt{(r_0y)}\,.\sqrt{(r_0z)}\,.\sqrt{(r_\lambda x)}\,.\sqrt{(r_\mu x)}\,.\sqrt{(r_\nu x)} = 0,$$

and

$$\Xi_{yz} = \Xi_{zx} = \Xi_{xy},$$

where

$$\Xi_{yz} = \frac{(r_0z)^2.\sqrt{(ay)^4} - (r_0y)^2.\sqrt{(az)^4}}{(yz)\,.\,(r_0y)(r_0z)}.$$

As to the addition-theorems of the Θ-functions, we evidently have in Quartic, §3, No. 26, a correlate to the sixteen addition-theorems of Weierstrass, belonging to our first family (I). (See *Am. Journal*, Vol. XVI, pp. 160, 161.) In the same way the nine families of the second type (II) have simple correlates, whereas to the six families of the third type (III) corresponds no equally simple result. We do not insist upon the general case, but we may point out the algebraic transformation corresponding to the transformation

$$\begin{aligned} u &= \frac{u_1 + v_1}{2}, & u_1 &= u + v, \\ v &= \frac{u_1 - v_1}{2}, & v_1 &= u - v. \end{aligned} \tag{30}$$

Denoting the binary variables corresponding to u_1 and v_1 by means of our formulæ (3) by ξ and η (whereas x and y correspond to u and v), we find

$$\left.\begin{aligned}(xy) &= -\sqrt{(r_0\xi)}\sqrt{(r_0\eta)},\\ (lx)(ly) &= -\sqrt{(r_\lambda\xi)}\sqrt{(r_\lambda\eta)},\\ (mx)(my) &= -\sqrt{(r_\mu\xi)}\sqrt{(r_\mu\eta)},\\ (nx)(ny) &= -\sqrt{(r_\nu\xi)}\sqrt{(r_\nu\eta)},\end{aligned}\right\}\quad(31)$$

$$\left.\begin{aligned}\sqrt{(ax)^4}.\sqrt{(ay)^4} &= -\sqrt[4]{G}.(\xi\eta),\\ (lx)^2.(ly)^2 &= -\sqrt{e_\mu-e_\nu}.(l\xi)(l\eta),\\ (mx)^2.(my)^2 &= -\sqrt{e_\nu-e_\lambda}.(m\xi)(m\eta),\\ (nx)^2.(ny)^2 &= -\sqrt{e_\lambda-e_\mu}.(n\xi)(n\eta),\end{aligned}\right\}\quad(32)$$

$$\left.\begin{aligned}2(ax)^4.(hy)^4 &= -\sqrt{G}.\{(a\xi)^2(a\eta)^2+\sqrt{(a\xi)^4}.\sqrt{(a\eta)^4}\},\\ 2(hx)^4.(ay)^4 &= -\sqrt{G}.\{(a\xi)^2(a\eta)^2-\sqrt{(a\xi)^4}.\sqrt{(a\eta)^4}\},\end{aligned}\right\}\quad(33)$$

$$\left.\begin{aligned}(tx)^6.(ay)^4.\sqrt{(ay)^4} &= \sqrt{G}.\sqrt[4]{G}.\{(a\eta)^3(a\xi).\sqrt{(a\xi)^4}-(a\xi)^3(a\eta).\sqrt{(a\eta)^4}\},\\ (ty)^6.(ax)^4.\sqrt{(ax)^4} &= \sqrt{G}.\sqrt[4]{G}.\{(a\eta)^3(a\xi).\sqrt{(a\xi)^4}+(a\xi)^3(a\eta).\sqrt{(a\eta)^4}\},\end{aligned}\right\}\quad(34)$$

$$\left.\begin{aligned}(hx)^4.(hy)^4 &= -\sqrt{G}.\{(h\xi)^2(h\eta)^2+\tfrac{1}{12}g_2.(\xi\eta)^2\},\\ (hx)^4.(hy)^4+\tfrac{1}{12}g_2.(ax)^4.(ay)^4 &= -\sqrt{G}.(h\xi)^2(h\eta)^2,\end{aligned}\right\}\quad(35)$$

$$(tx)^6.(ty)^6 = 2\sqrt{G}.\sqrt[4]{G}.(t\xi)^3(t\eta)^3,\quad(36)$$

etc.

Putting $v_1=0$, that is to say,

$$2u = u_1,\quad(37)$$

we obtain the following special formulæ, expressing the *duplication of the argument:*

$$\left.\begin{aligned}\sqrt{(ax)^4} &= -\sqrt[8]{G}.\sqrt{(r_0\xi)},\\ (lx)^2 &= -\sqrt[4]{e_\mu-e_\nu}.\sqrt{(r_\lambda\xi)},\\ (mx)^2 &= -\sqrt[4]{e_\nu-e_\lambda}.\sqrt{(r_\mu\xi)},\\ (nx)^2 &= -\sqrt[4]{e_\lambda-e_\mu}.\sqrt{(r_\nu\xi)},\end{aligned}\right\}\quad(38)$$

$$(hx)^4 = -\frac{1}{\sqrt[4]{G}}.(tr_0)^5(t\xi) = \sqrt[4]{G}.\sqrt{-\{(hr_0)^2(h\xi)^2+\tfrac{1}{12}g_2.(r_0\xi)^2\}},\quad(39)$$

$$(tx)^6 = 2\sqrt{G}\sqrt[8]{G}.\sqrt{(r_\lambda\xi)}\sqrt{(r_\mu\xi)}\sqrt{(r_\nu\xi)} = -\sqrt[4]{G}\sqrt[8]{G}\sqrt{2(tr_0)^3(t\xi)^3},\quad(40)$$

$$(tx)^6.\sqrt{(ax)^4} = \sqrt{G}.\sqrt[4]{G}.\sqrt{(a\xi)^4},\text{ etc.}\quad(41)$$

The transformations derived here from the theory of elliptic functions offer themselves naturally from the mere algebraic standpoint too. We are led to the formulæ (31) by a comparison of the identity

$$(xy)^2+[(lx)(ly)]^2+[(mx)(my)]^2+[(nx)(ny)]^2=0$$

with the identity

$$(r_0\xi)(r_0\eta)+(r_\lambda\xi)(r_\lambda\eta)+(r_\mu\xi)(r_\mu\eta)+(r_\nu\xi)(r_\nu\eta)=0.$$

(See Quartic No. 24, 28, 71.)

Finally, we mention a curious relation referring to four arguments, the sum of which is a period. Supposing

$$u_0+u_1+u_2+u_3\equiv 0 \bmod 2\bar{\omega}, \tag{42}$$

and denoting the x corresponding to the value u_κ of u by x_κ, we have

$$\begin{aligned}(x_2x_3)(x_3x_1)(x_1x_2)\,.\,\sqrt{(ax_0)^4}-(x_3x_0)(x_0x_2)(x_2x_3)\,.\,\sqrt{(ax_1)^4}\\ +(x_0x_1)(x_1x_3)(x_3x_0)\,.\,\sqrt{(ax_2)^4}-(x_1x_2)(x_2x_0)(x_0x_1)\,.\,\sqrt{(ax_3)^4}=0.\end{aligned} \tag{43}$$

Namely, under the said condition the determinant of the four functions Θ occurring in (8) and (9) vanishes; developing this determinant, we obtain the theorem.

Replacing the condition (42) by the ampler condition

$$u_0+u_1+u_2+u_3\equiv 0 \bmod \bar{\omega}, \tag{44}$$

and denoting, for sake of shortness, the polar $(tx_\iota)^3(tx_\kappa)^3$ by $t_{\iota\kappa}$, we have further

$$\begin{aligned}&(x_2x_3)(x_3x_1)(x_1x_2)\,.\,t_{23}\,.\,t_{31}\,.\,t_{12}\,.\,t_{00}\,.\,\sqrt{(ax_0)^4}\\ -&(x_3x_0)(x_0x_2)(x_2x_3)\,.\,t_{30}\,.\,t_{02}\,.\,t_{23}\,.\,t_{11}\,.\,\sqrt{(ax_1)^4}\\ +&(x_0x_1)(x_1x_3)(x_3x_0)\,.\,t_{01}\,.\,t_{13}\,.\,t_{30}\,.\,t_{22}\,.\,\sqrt{(ax_2)^4}\\ -&(x_1x_2)(x_2x_0)(x_0x_1)\,.\,t_{12}\,.\,t_{20}\,.\,t_{01}\,.\,t_{33}\,.\,\sqrt{(ax_3)^4}=0.\end{aligned} \tag{45}$$

The last addition-theorem is obtained by developing the vanishing determinant

$$|h_0^2,\ h_1f_1,\ f_2^2,\ t_3\sqrt{f_3}|.$$

The two addition-theorems (43) and (45) are transformed into each other by means of the transformation defined by (37)–(41). Namely, from these formulæ, or directly from (17) and (16), follows

$$2\,(xx')\,.\,(tx)^3(tx')^3=\sqrt{G}\,.\,\sqrt[4]{G}\,.\,(\xi\xi'), \tag{46}$$

x, x' and ξ, ξ' corresponding to one another by means of (37)–(41). By their substitution (45) and (44) are reduced to (43) and (42).

Specializing our formulæ by means of the supposition $u_0 = 0$, we obtain again the addition-theorems communicated on p. 225.

§6. *The Elementary Integrals.*

When in the formula (46) x and x' are brought close together by means of the supposition $u' = u + du$, we obtain, paying regard to (16) and (8),

$$-\frac{(xx')}{\sqrt[4]{f_x f_{x'}}}\cdot = -\tfrac{1}{2}\frac{(\xi\xi')}{\sqrt[4]{f_\xi f_{\xi'}}} \cong du = \tfrac{1}{2}\,du_1. \tag{47}$$

Replacing here x' and ξ' by $x + dx$ and $\xi + d\xi$, we obtain immediately the following expression of the integral of the first kind u:

$$\int_x^{r_0}\frac{(zdz)}{\sqrt{f_z}} = \tfrac{1}{2}\int_\xi^{r_0}\frac{(zdz)}{\sqrt{f_z}} \cong u.^* \tag{48}$$

The path of integration, of course, has to be chosen so that when z passes from r_0 to x, u varies from 0 to u, and not merely to a congruent value $u + 2\bar{\omega}$.

---

* Herewith the factor of proportionality, mentioned in the footnote on p. 217, is determined.

Replacing the formulæ (8) in which, as we have said already, only u, but not x, can be considered as an independent argument, by

$$\sqrt{\rho(x).(r_\kappa x)} \cong \Theta_\kappa(u) \qquad (\kappa = 0, \lambda, \mu, \nu),$$

and considering here the binary variables as independent quantities, we have defined in this way a homogeneous function $\rho(x)$ of the degree -1. Instead of (8) we obtain now

$$\rho^2(x).\sqrt{f_x} \cong -\Theta'.\Theta(2u),$$

whereas (48) does not change its form.

Consequently we have

$$\rho(x) = \frac{1}{\sqrt[4]{f_x}}\cdot\sqrt{\Theta'.\Theta\left\{2\int_{r_0}^{x}\frac{(zdz)}{\sqrt{f_z}}\right\}}.$$

Adding this factor to every pair of binary variables we transform all our formulæ into identities. The binary variables may now be considered as independent quantities. But only their ratio continues to enter into our formulæ; hence when u is given, the formulæ fail to determine the binary variables completely, as the simpler formulæ used in the text actually do.

In a similar way the elementary integrals of the second and third kind are calculated. We find

$$\begin{aligned}\frac{1}{2\sqrt{G}}\int_x^y \frac{(zdz)}{\sqrt{f_z}}\cdot\frac{(tr_\kappa)^5(tz)}{(r_\kappa z)} &\cong -\int_u^v dw\,\wp\,(u+\omega_\kappa) = \\ = \frac{\Theta_\kappa' v}{\Theta_\kappa v} - \frac{\Theta_\kappa' u}{\Theta_\kappa u} &= \zeta_\kappa v - \zeta_\kappa u \quad \begin{pmatrix}\kappa = 0,\ \lambda,\ \mu,\ \nu; \\ \omega_0 = 0;\ \Theta_0 = \Theta,\ \zeta_0 = \zeta\end{pmatrix}\end{aligned} \tag{49}$$

$$-2\int_x^y \frac{(zdz)}{\sqrt{f_z}}\cdot\frac{(hz)^4}{(az)^4} = \frac{1}{\sqrt{G}}\int_\xi^v \frac{(zdz)}{\sqrt{f_z}}\cdot\frac{(tr_0)^5(tz)}{(r_0 z)} \cong -2\int_u^v dw\,\wp\,(2w) = \zeta(2v) - \zeta(2u) \tag{50}$$

$$-2\int_x^y \frac{(zdz)}{\sqrt{f_z}}\cdot\left\{\frac{f_z}{l_z^2} - e_\lambda\right\} \cong \zeta_\lambda\,(2v) - \zeta_\lambda\,(2u), \tag{50b}$$

and further,

$$\int_{x_1}^{x_2} \frac{(xdx)}{\sqrt{f_x}}\cdot\frac{(ax)^2(ay)^2}{(xy)^2} \cong \begin{cases}\ \ \zeta\,(u_2+v) + \zeta\,(u_2-v) - \\ -\zeta\,(u_1+v) - \zeta\,(u_1-v),\end{cases} \tag{51}$$

$$\int_{x_1}^{x_2} \frac{(xdx)}{\sqrt{f_x}}\cdot\frac{(al)^2(ax)^2.(ly)^2 - (ax)^2(ay)^2}{[(lx)(ly)]^2} \cong \begin{cases}\ \ \zeta_\lambda\,(u_2+v) + \zeta_\lambda\,(u_2-v) - \\ -\zeta_\lambda\,(u_1+v) - \zeta_\lambda\,(u_1-v).\end{cases} \tag{51b}$$

Integrating once more, this time with respect to the parameter v, we obtain

$$\begin{gathered}-\int_{x_1}^{x_2}\int_{y_1}^{y_2} \frac{(xdx)}{\sqrt{f_x}}\cdot\frac{(ydy)}{\sqrt{f_y}}\cdot\frac{(ax)^2(ay)^2}{(xy)^2} \\ \cong \lg\frac{\Theta\,(u_1+v_1)\,\Theta\,(u_2+v_2)}{\Theta\,(u_1-v_1)\,\Theta\,(u_2-v_2)}\cdot\frac{\Theta\,(u_1-v_2)\,\Theta\,(u_2-v_1)}{\Theta\,(u_1+v_2)\,\Theta\,(u_2+v_1)};\end{gathered} \tag{52}$$

the corresponding formula (52b) may be omitted. Further we have

$$-\sqrt{f_y}\cdot\int_{x_1}^{x_2} \frac{(xdx)}{(xy)^2} \cong \begin{Bmatrix}\ \ \zeta\,(u_2+v) - \zeta\,(u_2-v) \\ -\zeta\,(u_1+v) + \zeta\,(u_1-v)\end{Bmatrix} = \frac{\partial}{\partial v}\lg\frac{\wp u_2 - \wp v}{\wp u_1 - \wp v}, \tag{53}$$

$$-\sqrt{f_y}\cdot\int_{x_1}^{x_2} \frac{(xdx)}{[(lx)(ly)]^2} \cong \begin{Bmatrix}\ \ \zeta_\lambda(u_2+v) - \zeta_\lambda(u_2-v) \\ -\zeta_\lambda(u_1+v) + \zeta_\lambda(u_1-v)\end{Bmatrix} = \frac{\partial}{\partial v}\lg\frac{\wp u_2 - \wp\,(v+\omega_\lambda)}{\wp u_1 - \wp\,(v+\omega_\lambda)}. \tag{53b}$$

Integrating once more with respect to the parameter, we obtain

$$\begin{aligned}&\int_{x_1}^{x_2}\int_{y_1}^{y_2} \frac{(xdx)(ydy)}{(xy)^2} = \lg\frac{(x_1y_1)(x_2y_2)}{(x_1y_2)(x_2y_1)} \cong -\int_{u_1}^{u_2}\int_{v_1}^{v_2} du\,dv\,[\wp\,(u+v) - \wp\,(u-v)] \\ &= -\tfrac{1}{2}\lg\frac{\wp\,(u_1+v_1) - \wp\,(u_1-v_1)}{\wp\,(u_1+v_2) - \wp\,(u_1-v_2)}\cdot\frac{\wp\,(u_2+v_2) - \wp\,(u_2-v_2)}{\wp\,(u_2+v_1) - \wp\,(u_2-v_1)} \\ &= \lg\frac{\wp u_1 - \wp v_1}{\wp u_1 - \wp v_2}\cdot\frac{\wp u_2 - \wp v_2}{\wp u_2 - \wp v_1} = \lg\frac{\Theta\,(u_1+v_1)\,\Theta\,(u_1-v_1)}{\Theta\,(u_1+v_2)\,\Theta\,(u_1-v_2)}\cdot\frac{\Theta\,(u_2+v_2)\,\Theta\,(u_2-v_2)}{\Theta\,(u_2+v_1)\,\Theta\,(u_2-v_1)},\end{aligned} \tag{54}$$

and similarly

$$\int_{x_1}^{x_2}\int_{y_1}^{y_2}\frac{(xdx)(ydy)}{[(lx)(ly)]^2}=\lg\frac{(lx_1)(ly_1)\,.\,(lx_2)(ly_2)}{(lx_1)(ly_2)\,.\,(lx_2)(ly_1)}\cong$$
$$\cong\lg\frac{\Theta_\lambda(u_1+v_1)\,\Theta_\lambda(u_1-v_1)}{\Theta_\lambda(u_1+v_2)\,\Theta_\lambda(u_1-v_2)}\cdot\frac{\Theta_\lambda(u_2+v_2)\,\Theta_\lambda(u_2-v_2)}{\Theta_\lambda(u_2+v_1)\,\Theta_\lambda(u_2-v_1)}. \tag{54b}$$

Hence comparing (51) and (53), we obtain

$$\int_{x_1}^{x_2}\frac{(xdx)}{\sqrt{f_x}}\cdot\frac{(ax)^2(ay)^2\mp\sqrt{f_x}\sqrt{f_y}}{2\,(xy)^2}\cong\zeta(u_2\pm v)-\zeta(u_1\pm v), \tag{55}$$

and a similar equation is obtained by comparison of (51b) and (53b); and comparing (52) and (54), we have

$$\mp\int_{x_1}^{x_2}\int_{y_1}^{y_2}\frac{(xdx)}{\sqrt{f_x}}\cdot\frac{(ydy)}{\sqrt{f_y}}\cdot\frac{(ax)^2(ay)^2\mp\sqrt{f_x}\sqrt{f_y}}{2\,(xy)^2}\cong$$
$$\cong\lg\frac{\Theta(u_1\pm v_1)\,\Theta(u_2\pm v_2)}{\Theta(u_1\pm v_2)\,\Theta(u_2\pm v_1)},\ \text{etc.} \tag{56}$$

We denote the double integral on the left, adopting the *lower* sign, by $Q\begin{bmatrix}x_2y_2\\x_1y_1\end{bmatrix}$, and write (56) thus:

$$Q\begin{bmatrix}x_2y_2\\x_1y_1\end{bmatrix}\cong\lg\frac{\Theta(u_1-v_1)\,\Theta(u_2-v_2)}{\Theta(u_1-v_2)\,\Theta(u_2-v_1)},$$
$$Q\begin{bmatrix}x_2\bar{y}_2\\x_1\bar{y}_1\end{bmatrix}\cong\lg\frac{\Theta(u_1+v_1)\,\Theta(u_2+v_2)}{\Theta(u_1+v_2)\,\Theta(u_2+v_1)}, \tag{56}$$

the replacement of y by $\bar{y}$ indicating that the system of values y, $\sqrt{f_y}$ is replaced by the *conjugate* system of values y, $-\sqrt{f_y}$.

An easy specialization of our formulæ furnishes now expressions of the functions $\zeta_\kappa u$, etc., themselves in terms of covariant integrals: *We have simply to introduce conjugate limits in the above formulæ.* Thus we obtain

$$Z_\kappa(x)=-\frac{1}{4\sqrt{G}}\int_x^{\bar{x}}\frac{(zdz)}{\sqrt{f_z}}\cdot\frac{(tr_\kappa)^5(tz)}{(r_\kappa z)}\quad\cong\zeta_\kappa u\qquad(\kappa=0,\ \lambda,\ \mu,\ \nu), \tag{57}$$

$$\mathfrak{Z}(x)=\int_x^{\bar{x}}\frac{(zdz)}{\sqrt{f_z}}\cdot\frac{h_z}{f_z}\quad\cong\zeta(2u), \tag{58}$$

$$\mathfrak{Z}_\lambda(x)=\int_x^{\bar{x}}\frac{(zdz)}{\sqrt{f_z}}\cdot\left\{\frac{f_z}{l_z^2}-e_\lambda\right\}\cong\zeta_\lambda(2u), \tag{58b}$$

$$-\tfrac{1}{4}\left\{\int_x^{\bar{x}}\frac{(zdz)}{\sqrt{f_z}}\cdot\frac{(ay)^2(az)^2}{(yz)^2}+\int_y^{\bar{y}}\frac{(zdz)}{\sqrt{f_z}}\cdot\frac{(ax)^2(az)^2}{(xz)^2}\right\}=Z(x,\,y)\cong\zeta(u+v);$$
$$Z(x,\,\bar{y})\cong\zeta(u-v), \tag{59}$$

etc. The mutual relations of the functions $Z_\kappa(x)$ and $Z(x, y)$ introduced here are evident. As to $\mathfrak{Z}(x)$, we notice that $Z(x, y)$ remains finite and definite when y coincides with x. $Z(x, x) = \lim\limits_{y=x} Z(x, y)$ represents, however, an infinity of different analytic functions of x, corresponding to the different suppositions $v = u + 2\bar{\omega}$. One of them is our $\mathfrak{Z}(x)$; it corresponds to the simplest assumption $v = u$, that is to say, to the supposition that in the integrals contained in (59) not only the limits, but also the paths of integration are brought close together.

A similar remark holds with respect to the function $Q\begin{bmatrix} x_2 \bar{y}_2 \\ x_1 \bar{y}_1 \end{bmatrix}$ defined in No. (56). We denote the function of x_1, x_2, which results when v_1 coincides with u_1 and v_2 with u_2, by $Q\binom{x_2}{x_1}$. Then we have

$$Q\binom{y}{x} \cong \lg \frac{\Theta(2u)\,\Theta(2v)}{\Theta^2(u+v)};$$

on the other hand, according to (8) and (16),

$$\frac{(xy)}{\sqrt[4]{f_x f_y}} \cong \frac{\Theta(u+v)\,\Theta(u-v)}{\Theta'.\sqrt{\Theta(2u)\,\Theta(2v)}},$$

consequently

$$\begin{aligned} \frac{(xy)}{\sqrt[4]{f_x f_y}}\,.\,e^{\frac{1}{2}Q\binom{y}{x}} &\cong \frac{\Theta(u-v)}{\Theta'} = \sigma(u-v), \\ \sqrt{-1}\,.\,\frac{(xy)}{\sqrt[4]{f_x f_y}}\,.\,e^{\frac{1}{2}Q\binom{\bar{y}}{x}} &\cong \frac{\Theta(u+v)}{\Theta'} = \sigma(u+v); \end{aligned} \tag{60}$$

further, denoting the products $\sqrt{(r_0 x)}\sqrt{(r_\lambda x)}$ and $\sqrt{(r_\mu x)}\sqrt{(r_\nu x)}$ by $\sqrt{\psi_x}$ and $\sqrt{\chi_x}$, on account of (18),

$$\begin{aligned} -\frac{\sqrt{\psi_x}\sqrt{\chi_y}+\sqrt{\psi_y}\sqrt{\chi_x}}{\sqrt[4]{f_x f_y}}\,.\,e^{\frac{1}{2}Q\binom{y}{x}} &\cong \frac{\Theta_\lambda(u-v)}{\Theta_\lambda} = \sigma_\lambda(u-v), \\ -\sqrt{-1}\,\frac{\sqrt{\psi_x}\sqrt{\chi_y}-\sqrt{\psi_y}\sqrt{\chi_x}}{\sqrt[4]{f_x f_y}}\,.\,e^{\frac{1}{2}Q\binom{\bar{y}}{x}} &\cong \frac{\Theta_\lambda(u+v)}{\Theta_\lambda} = \sigma_\lambda(u+v). \end{aligned} \tag{60b}$$

The greater part of the preceding results is not new. Especially the formulæ (56), (60) and (60b) have been studied by F. Klein and H. Burkhardt (Math. Annalen, v. 27, 32), after Klein, paying regard mainly to what we have called the general distribution of the argument, had introduced the first formulæ in (60) and (60b) as definitions of the σ-functions. There are, however, some difficulties connected with these expressions which seem to make further investigations desirable, notwithstanding the valuable researches of Burkhardt. The function $Q\binom{y}{x}$ is not so easy to deal with as one might wish on account of its central position in the theory, apparently for the reason of its definition as the limit of a double integral. We have not been able to get rid of these complications without introducing others. Only one expression, of $\sigma(2u)$, may therefore be mentioned, which is remarkable for the part the covariant Φ (Quartic No. 36) plays in it:

$$\sigma(2u) \cong \frac{\sqrt{f_x}}{\sqrt[3]{-\frac{T}{2}}} \cdot e^{-6\int_{r_0}^{x} \frac{(ydy)}{\sqrt{f_y}} \int_{y}^{\bar{y}} (zdz) \cdot \frac{\sqrt{f_z} \cdot \Phi_z}{T_z^2}} . \tag{61}$$

T is the sextic covariant, $= t$.

§7. *Additional Remarks concerning the Transformation of some Integrals.*

The results hitherto developed establish the connection between two series of analytical expressions: The homogeneous functions on the left side of our formulæ refer to the general case of the binary quartic, the functions on the right to Weierstrass' canonical form, and to the special shape the theory of elliptic functions assumes when this form is made the basis of the calculations. It is easy, of course, to specialize our considerations in such a way that any given canonical form of the quartic appears on the left. Thus we might investigate the connection among different shapes of the theory, based upon the consideration of a binary quartic, without losing the general points of view furnished by the theory of invariants. It is not our intention to dwell upon these details; we may call attention, however, to a certain canonical form of the quartic which has not yet been considered by previous authors, as far as we know.

Its properties are, in a certain respect, opposite to those of Weierstrass' canonical form, and it seems to offer some advantage when not only the argument u, but also the periods ω, ω' are considered as variables. In this case in Weierstrass' canonical form three of the vanishing points of the quartic undergo a change of situation, whereas the last one, the point infinity, alone rests fixed. But we might just as well keep three points in fixed positions and permit the last one alone to vary. The fixed points may be any three arbitrarily chosen points of the binary domain; for instance, the three cube roots of unity, or the points 0, 1, ∞ ("Riemann's canonical form," see Klein's Modulfunctionen, I, p. 25). Instead of specializing the fixed points in this way, we prefer to consider them as the vanishing points of a given but entirely *arbitrary cubic*, and to identify the corresponding linear factors with the irrational covariants (λx), (μx), (νx) of this cubic. *Thus the canonical form in question is characterized by the fact that the sum of three linear factors of the quartic vanishes identically.*

We have two different modes of transforming a given quartic into this canonical form which can evidently be reached only through irrational operations. The first consists simply in replacing the products $(r_\mu r_\nu).(r_\lambda x)$, $(r_\nu r_\lambda).(r_\mu x)$, $(r_\lambda r_\mu).(r_\nu x)$, by (λx), (μx), (νx) respectively; it is a linear transformation. The second way is shown by the theory of the octahedron, developed in "Quartic," §8. We shall confine ourselves to the second case, considering the transformation of the elliptic integral of the first kind.

Writing the formulæ (59)–(61) on p. 192 once more, with a second pair of corresponding variables η, y (instead of ξ, x), we find, after a short calculation,

$$\sqrt{-r}.(\xi\eta) \cong 4\sqrt{-R}.(xy).(tx)^3(ty)^3; \tag{62}$$

consequently, when η and y nearly coincide with ξ and x,

$$\sqrt[4]{-r}.\frac{(\xi d\xi)}{\sqrt{p}} \cong 4\sqrt[4]{-R}.(xdx). \tag{63}$$

On the other hand, denoting the sum $e_\lambda.(\lambda\xi) + e_\mu(\mu\xi) + e_\nu(\nu\xi)$ by $3(\rho\xi)$, we have

$$\sqrt{-r}.(\rho\xi) \cong \sqrt{-R}.(ax)^4. \tag{64}$$

Combining the last pair of formulæ and choosing r_0 as the lower limit of the integral referring to the variable x, we obtain the transformation in question:

$$\int_\rho^\xi \frac{(\xi d\xi)}{2\sqrt{p}.\sqrt{(\rho\xi)}} \cong 2\int_{r_0}^x \frac{(xdx)}{\sqrt{f}}. \tag{65}$$

The interest of the formula consists mainly in the decomposition of the elliptic radical into a cubic and a linear factor, *the first of which is entirely independent of the quartic f.*

The equality (65) is another expression of the one-to-four correspondence between ξ and x established by the formulæ (59)–(61) on p. 192. Consequently the two corresponding periods of integrals in (65) must be *equal*, and the invariants of f must be equal to the corresponding invariants of the quartic $4p.(\rho\xi)$. This is actually the case, as a simple calculation shows. *Thus we see that our principle of transference, by which the covariants of a cubic are transformed into the covariants of the octahedron, represents a peculiar interpretation of the duplication of the argument in the theory of elliptic functions.*

Although it does not exactly belong to our present topic, we may mention that the same principle of transference furnishes still some other remarkable transformations of integrals.

We have, for instance, on account of (63):

$$\sqrt[3]{-r}\int\frac{(\xi d\xi)}{p^{\frac{2}{3}}}\cong 4\sqrt[3]{-R}\int\frac{(xdx)}{T^{\frac{1}{3}}},$$

$$\sqrt{-r}\int\frac{(\xi d\xi)}{q^{\frac{2}{3}}}\cong 4\sqrt{-R}\int(xdx).\frac{T}{\Psi^{\frac{2}{3}}}.$$

By means of these formulæ the integrals on the right are recognized as elliptic integrals, belonging to the equianharmonic case (the case in which complex multiplication by a cube root of unity takes place). The reduction of the first integral is known; it is due to Brioschi (Sulla equazione dell' ottaedro, Trans. della R. Acc. dei Lincei, ser. III, v. III, p. 233).

Another transformation worth mentioning is the following one:

$$(-r)^{\frac{5}{6}}\int\frac{(\xi d\xi)(\rho\xi)}{\sqrt{p}\sqrt{q}}\cong 4\,(-R)^{\frac{5}{6}}\int(xdx).\frac{(ax)^4}{\sqrt{\Psi}}.$$

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Semi-Combinants as Concomitants of Affiliants.

BY HENRY S. WHITE.

INTRODUCTION.

The theory of invariants received its natural extension and completion in that of seminvariants. So from the study of combinants it would be expected that Sylvester should have advanced to the investigation of semi-combinants. Since, however, a combinant differs from other invariants only in being invariant with respect to two or more independent systems of linear transformations, such an extension promised little novelty of method. I allude to this possibility only in order to propose a larger use for the term *semi-combinant.* In Sylvester's original formulation (1853) combinants are defined as "concomitants to systems of functions remaining invariable, not only when combinations of the variables are substituted for the variables, but also when combinations of the functions are substituted for the functions."* These have their place in the discussion of a system of forms or quantics, all of which have the same order in the variables. Where, however, a system of forms of unequal orders is to be discussed, there too arise concomitants which belong to the "system in its corporate capacity." But linear combination of the forms with constant multipliers is not possible, since homogeneity in the variables must be preserved. Accordingly the "system in its corporate capacity" will comprise compound forms, each form being modified by the addition of each form of lower order than itself, multiplied by an arbitrary form of suitable order. The coefficients of each such arbitrary form constitute the constants of an independent transformation. An invariant of one such transformation I propose to call a *semi-combinant* of the two forms concerned; an invariant of all such transformations, a *semi-combinant of the entire system.*

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* Cambridge and Dublin Math. Jour., Vol. VIII, p. 62.

For example, let the groundforms be two binary forms of orders m and n respectively, f_m and ϕ_n, where $m > n$. Borrowing Sylvester's terminology, let us call the compound form

$$F_m = f_m + R_{m-n} \cdot \phi_n$$

the conjunctive* of the system. The arbitrary constants are then the $m-n+1$ coefficients of the arbitrary form R_{m-n}. Then every concomitant of the conjunctive F_m and the lower form ϕ_n will be a semi-combinant of f_m and ϕ_n, if it is independent of the arbitrary constants in R_{m-n}.

It is possible that the arbitrary form R_{m-n} can be equated to such a covariant as to make conjunctive F itself a semi-combinant. If this be done I call the particular value of the conjunctive a *semi-combinant groundform*. Having defined the term, we must inquire whether semi-combinant groundforms exist, and if so, how many there are that are linearly independent. I shall prove that if such a groundform exists it must satisfy identically a definite equation of condition found by elimination from a set of linear equations, and invariant in structure; an invariant equation, of which the well-known condition that a curve shall be apolar to a conic is a special example. Reversing now the order of things, I consider all groundforms that are included in the conjunctive of the system, and those of them that satisfy invariant equations of suitable order, linear in their coefficients, I designate as *affiliant* groundforms. I show that not only is every semi-combinant groundform an affiliant, but also every affiliant groundform is a semi-combinant. A given characteristic equation resolves into linear equations which determine the corresponding affiliant groundform, thus establishing directly the existence of semi-combinant groundforms. That their number is limited I show from the nature of the characteristic equations on the one hand; on the other hand, from the necessary structure of those groundforms as covariants. Both methods give the same upper limit, probably much too high, to their number. So much constitutes the main purpose of this paper.

Incidentally it will be noticed that apolarity is, in the binary domain and for quadric forms in any number of variables, a special case of the affiliant relation. Indeed the well-known connection between the theory of apolarity and the theory of combinants† finds its analogue in the relation of affiliants to semi-combinants. The range of possible applications of this general theory is as wide as could be wished, since the simultaneous system may include any number whatever of groundforms in any number of variables, of any orders, alike or different. That nearest at hand is probably the application to curves of double curvature. Toward this I offer elementary suggestions; and similarly with regard to the fundamental question concerning a reduced form-system of semi-combinants. Finally, I point out in what way the introduction of semi-combinants may be useful in the discovery of normal forms, in special cases, for Abelian integrals, and suggest the extension of the theory to connexes and other mixed forms.

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* Cambridge and Dublin Math. Jour., Vol. VIII, pp. 258, 259.

† See for example B. Igel: Ueber einige Anwendungen des Principes der Apolarität, Wiener Berichte XCII, pp. 1153-1194. Full references are given by F. Meyer in Jahresbericht der Deutschen Mathematiker-Vereinigung, Bd. I, p. 254 seq.

§1. *Covariant Curve apolar to a Conic determined as a Semi-Combinant.*

To Rosanes and Reye is due principally the theory of mutually apolar curves or surfaces. Rosanes called such curves *conjugirt* in relation to each other. Lindemann later used the still less distinctive appellation *vereinigt liegend;* but the term *apolar,* proposed by Reye, seems to be in use at present to the exclusion of both the others. Its meaning may be defined briefly in algebraic language as follows. If a locus of order n and a locus of class k be given by the equations

$$a_x^n = 0 \text{ and } u_\alpha^k = 0,$$

these loci are mutually apolar when the covariant of order $n-k$ (or contravariant of class $k-n$) linear in the coefficients of both equations is identically zero; i. e. when

$$a_\alpha^k a_x^{n-k} \equiv 0 \text{ if } n \geqq k,$$

or

$$a_\alpha^n u_\alpha^{k-n} \equiv 0 \text{ if } k \geqq n,$$

for all values of the variables (x) or (u). Inasmuch as each locus has both order and class, Reye disclosed an important theorem when he proved that the apolar relation is reciprocable; or, in other words, that if two loci are apolar, their polar reciprocals also are apolar.[*]

The study of apolarity received fresh impetus when Lindemann[†] found it available in the discussion of binary quantics by the aid of rational curves. A binary form of even order $2n$ is transformed readily into a ternary quantic of order n, and thus its zeros (*roots* of the corresponding equation), which are ordinarily represented by $2n$ points on a line, come to be represented by the $2n$ points where a conic is met by a curve of order n. The equations of transformation determine the equation of the conic, the transformed quantic equated to zero determines the intersecting n^{ic}. *Lindemann showed that this n^{ic} is always apolar to the conic.* Now if the conic were given arbitrarily, and the $2n$ points upon it, the problem of finding the corresponding binary $2n^{ic}$ would require the discovery of a curve of n^{th} order intersecting the conic in the given points, and apolar to the conic. While through the $2n$ points there pass not only one but an infinity of curves of order n, only one of them is, in general, apolar to the conic. Lindemann's theorem may be stated thus:

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[*] Th. Ruge: Ueber algebraische Flächen, die zu einander apolar sind, Journal für r. u. a. Mathematik, Bd. 79, pp. 159–175.

[†] F. Lindemann: Sur une représentation géométrique des covariants des formes binaires, Bull. S. M. F., t. V, pp. 113–126.

Every complete intersection-system of points on a conic determines one, and only one, curve apolar to the conic, meeting the conic in those points and in no others.

Representing the conic and any curve which meets it in the given complete intersection-system by the equations

$$a_x^2 = 0 \text{ and } \alpha_x^n = 0,$$

Lindemann determined the apolar n^{ic} through the intersection-system as a rational covariant of these two curves.* Its equation is necessarily of the form

$$A_x^n = \pi_0 . \alpha_x^n + \pi_x^{n-2} a_x^2 = 0, \tag{1}$$

where π_x^{n-2} is some quantic of order $n-2$, π_0 a constant. That A_x^n is a covariant is obvious, assuming that it is uniquely determinate, from the invariant character of its equation of condition, viz.

$$(abA)^2 A_x^{n-2} \equiv 0. \tag{2}$$

Instead of reproducing Lindemann's derivation of this covariant A_x^n, I will give a more rapid derivation by the aid of its semi-combinant property.

There is but one quantic A_x^n satisfying the equation of apolarity; for, substituting the form (1) in equation (2) we have

$$\frac{n(n-1)}{2} \pi_0 (ab\alpha)^2 \alpha_x^{n-2} + \frac{(n-2)(n-3)}{2} (ab\pi)^2 \pi_x^{n-4} c_x^2$$
$$+ 2(n-2)(ab\pi)(abc)\pi_x^{n-1} c_x + (abc)^2 \pi_x^{n-2} \equiv 0,$$

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* F. Lindemann: Sur une représentation géométrique des covariants des formes binaires, 2me note, Bull. S. M. F., t. VI, p. 195-208.

or after reduction of the third term,

$$\frac{n\,(n-1)}{2}\,\pi_0\,(aba)^2\,a_x^{n-2}+\frac{(n-2)(n-3)}{2}\,(ab\pi)^2\,\pi_x^{n-2}c_x^2$$
$$+\frac{2n-1}{3}\,(abc)^2\,\pi_x^{n-2}\equiv 0. \qquad (3)$$

On separating this identical equation into its constituents, the latter would be linear in π_0 and the coefficients of π_x^{n-2}. Hence they are solved by a single system of values, or else by an indefinite number. Obviously equation (3) gives a recurrent method for determining the quantic π_x^{n-2} with its first term $\frac{3n\,(n-1)}{2\,(2n-1)}\cdot\pi_0\cdot\frac{(aba)^2a_x^{n-2}}{(abc)^2}$ perfectly determinate. Accordingly, there is one, and only one, apolar quantic A_x^n.

This covariant A_x^n must be a semi-combinant of a_x^n and a_x^2. As it is unique, and its determining equation (2) is independent of the quantic a_x^n, it can depend only upon the given intersection-system, and that remains unchanged when the curve $a_x^n=0$ is replaced by any curve of the system

$$a_x^n+u_x^{n-2}.\,a_x^2=0,$$

where u_x^{n-2} denotes a quantic of order $n-2$ with independently variable coefficients. The covariant A_x^n remains unaltered therefore by the substitution of $a_x^n+u_x^{n-2}\,a_x^2$ for a_x^n; that is, the covariant A_x^n is a semi-combinant.

As this semi-combinant is linear in the coefficients of a_x^n, its terms can contain symbolic factors of only the following types:

$$a_x^2,\ (abc)^2,\ (aba)^2,\ a_x^{n-2i} \qquad (i=0,\,1,\,2,\,\ldots.).$$

This is readily seen by the aid of ordinary theorems on ternary quadrics. I may therefore write the covariant as follows, using undetermined coefficients λ_0, λ_1, λ_2, etc., and denoting by Δ the discriminant $(abc)^2$, by a_Δ^2 the factor $(aab)^2$:

$$A_x^n=\lambda_0\Delta^r.\,a_x^n+\lambda_1.\,\Delta^{r-1}.\,a_\Delta^2a_x^{n-2}.\,a_x^2+\lambda_2\Delta^{r-2}.\,a_\Delta^2a_\Delta^2a_x^{n-4}.\,a_x^2b_x^2+\text{etc.} \qquad (4)$$

The number of constants λ_k is $\frac{n+1}{2}$ or $\frac{n+2}{2}$, as n is odd or even. To determine their values, Lindemann applied the equation of apolarity, (2). I will employ the condition that renders A_x^n a semi-combinant. Substituting in (4) for

a_x^n the conjunctive $a_x^n + u_x^{n-2}a_x^2$, the increment of A_x^n must vanish identically. From the first term of (4) comes the increment

$$\lambda_0 \Delta^r u_x^{n-2} a_x^2 .$$

From the second,

$$\frac{2}{n\,(n-1)}\,\lambda_1 \Delta^{r-1}\left\{a_\Delta^2 u_x^{n-2} + 2\,(n-2)\,a_\Delta u_\Delta a_x u_x^{n-3} + \frac{(n-2)(n-3)}{1.2}\,.\,u_\Delta^2 u_x^{n-4} a_x^2\right\} b_x^2$$

which reduces to

$$\frac{2}{n\,(n-1)}\,\lambda_1\left\{\frac{2n-1}{3}\,.\,\Delta^r u_x^{n-2}a_x^2 + \frac{(n-2)(n-3)}{1.2}\,.\,\Delta^{r-1}.\,u_\Delta^2 u_x^{n-4}.\,a_x^2 b_x^2\right\}.$$

Similarly the third term gives the reduced increment,

$$\frac{2}{n\,(n-1)}\,\lambda_2 .\left\{\begin{array}{l}\left(1+1+\dfrac{2.2}{3}+\dfrac{4\,(n-4)}{3}\right)\Delta^{r-1}u_\Delta^2 u_x^{n-4}.\,a_x^2.\,b_x^2\\ \qquad + \dfrac{(n-4)(n-5)}{1.2}\,\Delta^{r-2}u_\Delta^2 u_{\Delta'}^2 u_x^{n-6} a_x^2 b_x^2 c_x^2\end{array}\right\}.$$

These suffice to show the law of the coefficients. We find on collecting terms and equating to zero,

$$\lambda_0 + \frac{2\,(2n-1)}{3n\,(n-1)}\,\lambda_1 = 0,$$

$$\lambda_1 + \frac{2.2.(2n-3)}{3\,(n-2)(n-3)}\,\lambda_2 = 0, \text{ etc.}$$

Hence

$$\lambda_1 = -\frac{3n\,(n-1)}{2\,(2n-1)}\,\lambda_0,$$

$$\lambda_2 = +\frac{3^2.\,n\,(n-1)(n-2)(n-3)}{2.2^2\,(2n-1)(2n-3)}\,\lambda_0 .$$

Introducing additional factors in both terms of these fractions, we obtain them in the form

$$\lambda_1 = -\frac{3n.\,n\,(n-1)}{2n.\,(2n-1)}\,\lambda_0 = -\frac{3\,{}_nP_1 .\,{}_nP_2}{1.\,{}_{2n}P_2}\,\lambda_0,$$

$$\lambda_2 = +\frac{3^2.\,{}_nP_2 .\,{}_nP_4}{2\,!\,{}_{2n}P_4}\,\lambda_0 .$$

From the general equation of condition we obtain thus the value of λ_k in terms of λ_0:

$$\lambda_k = (-1)^k\,\frac{3^k\,{}_nP_k\,{}_nP_{2k}}{k\,!\,{}_{2n}P_k}\,\lambda_0 . \tag{5}$$

Here for abbreviation ${}_nP_k$ denotes the product

$${}_nP_k = n(n-1)(n-2)\,\ldots\ldots\,(n-k+1).$$

The value of λ_0 is arbitrary; we may assume $\lambda_0 = 1$. Omitting needless factors, the value of r may be taken less by unity than the number of possible terms

$$r = \frac{n-1}{2} \text{ or } \frac{n}{2}.$$

Inserting these values of $\lambda_0 \ldots\ldots \lambda_k \ldots\ldots$, and r, in formula (4), we have the covariant and semi-combinant A_x^n expressed in terms of fundamental covariants of a_x^n and a_x^2.

The expression for A_x^n, obtained by its semi-combinant property, agrees with that derived by Lindemann from its apolarity to a_x^2. The foregoing discussion is a proof of Lindemann's theorem, since the semi-combinant property is in this case involved in the property of apolarity.

A precisely similar determination of a semi-combinant n^{ic} can be found when we interpret the forms a_x^n and a_x^2 as ternary, or quaternary, or m-ary. In each case the form obtained is identical with that given by the conditions of apolarity. The reason for this will appear in §§7 and 8 below.

§2. *The Differential Equations satisfied by Semi-Combinants.*

The name semi-combinant has been proposed above for certain invariants of a simultaneous system. Invariants (including, of course, covariants, etc.) are defined readily by differential equations which form a "complete system," and ordinary invariants thus come to be treated by Lie's method as invariants of a "group" of infinitesimal transformations. The same can be shown to be true of semi-combinants. In verification of this statement I need consider only two binary quantics as an example; it will appear that the discussion could be extended without difficulty to any number of quantics in any number of variables.

Denote by f_m and ϕ_n binary quantics of orders m and n respectively, and let $m > n$. To indicate with precision the several coefficients, let the terms of the two quantics be written without numerical factors,

$$f_m = \sum_{i+k=m} (a_{i,\,k} x_1^i x_2^k),$$

$$\phi_n = \sum_{i+k=n} (\alpha_{i,\,k} x_1^i x_2^k).$$

Then the four operators discussed in the theory of linear transformations are these:

$$\left.\begin{aligned}
W_1 &= \sum_{i+k=m}\left(i\,.\,a_{ik}\frac{\partial}{\partial a_{ik}}\right)+\sum_{i+k=n}\left(ia_{ik}\frac{\partial}{\partial a_{ik}}\right),\\
W_2 &= \sum_{i+k=m}\left(ka_{ik}\frac{\partial}{\partial a_{ik}}\right)+\sum_{i+k=n}\left(ka_{ik}\frac{\partial}{\partial a_{ik}}\right),\\
O_1 &= \sum_{i+k=m}\left((i+1)\,a_{i+1,\,k-1}\frac{\partial}{\partial a_{ik}}\right)+\sum_{i+k=n}\left((i+1)\,a_{i+1,\,k-1}\frac{\partial}{\partial a_{ik}}\right),\\
O_2 &= \sum_{i+k=m}\left((k+1)\,a_{i-1,\,k+1}\frac{\partial}{\partial a_{ik}}\right)+\sum_{i+k=n}\left((k+1)\,a_{i-1,\,k+1}\frac{\partial}{\partial a_{ik}}\right).
\end{aligned}\right\}\qquad(6)$$

These are here written as applied to invariants only; the variables neglected may be included, of course, by adjoining a linear quantic to the two under consideration. Of these four operators, the first two acting on an invariant reproduce it with a numerical factor, the third and fourth yield identically zero.

The additional operations entering into the theory of semi-combinants are educed from the definition. A semi-combinant of f_m and ϕ_n is an invariant which is unchanged when the coefficients of f_m are increased by the corresponding coefficients of $u_{m-n}\phi_n$; u_{m-n} denoting an arbitrary quantic of order $(m-n)$. Giving to u_{m-n} successively the $(m-n+1)$ linearly independent values λx_1^{m-n}, $\lambda x_1^{m-n-1}x_2$, ... λx_2^{m-n}, we shall obtain $(m-n+1)$ independent sets of increments to be applied to the coefficients of f_m, and shall find as many equations of condition for a semi-combinant.

To the coefficients

$$a_{m,0},\quad a_{m-1,1},\,\ldots\ldots\,a_{1,m-1}\;,\;a_{0,m}\;,$$

accrue the increments

$$\begin{array}{rllll}
(1) & \lambda a_{n,0}, & \lambda a_{n-1,1}, & \ldots\ldots\,0 & ,\;0\;, \\
(2) & 0 & ,\;\lambda a_{n-0} & ,\ldots\ldots & \\
 & \vdots & & & \\
(m-n) & 0 & ,\;0 & ,\ldots\ldots\,\lambda a_{0,n} & ,\;0\;, \\
(m-n+1) & 0 & ,\;0 & ,\ldots\ldots\,\lambda a_{1,n-1}, & \lambda a_{0,n}.
\end{array}$$

Corresponding to these elementary sets of increments, construct the $(m-n+1)$ elementary differential operators:

$$\left.\begin{aligned}
S_{m-n,\,0} &= a_{n,\,0}\frac{\partial}{\partial a_{m,\,0}} + a_{n-1,\,1}\frac{\partial}{\partial a_{m-1,\,1}} + \dots + a_{0,\,n}\frac{\partial}{\partial a_{m-n,\,n}} \\
&= \sum_{\left(\substack{i+k=n\\ l+k=m}\right)}\left(a_{ik}\frac{\partial}{\partial a_{lk}}\right) \\
S_{m-n-1,\,1} &= \sum\left(a_{i+1,\,k-1}\frac{\partial}{\partial a_{l,\,k}}\right), \\
S_{m-n-2,\,2} &= \sum\left(a_{i+2,\,k-2}\frac{\partial}{\partial a_{l,\,k}}\right), \\
&\vdots \\
S_{0,\,m-n} &= \sum\left(a_{i,\,k}\frac{\partial}{\partial a_{i,\,m-i}}\right).
\end{aligned}\right\} \qquad (7)$$

The sets of increments tabulated above define each a substitution of functions of an arbitrary λ for the coefficients a_{ik}, and these substitutions must not alter the value of a semi-combinant F. Equating to zero the corresponding increments of F, there result the equations of condition:

$$\left.\begin{aligned}
S_{m-n,\,0}(F) = 0,\quad S_{m-n-1,\,1}(F) = 0,\quad S_{m-n-2,\,2}(F) = 0,\\
S_{0,\,m-n}(F) \;= 0.
\end{aligned}\right\} \qquad (8)$$

Referring to the definitions (7), it is apparent upon inspection that any two operators S are commutative, for the action of either upon the other is nil. Since then for all suffices i, k,

$$S_{m-n-i,\,i}\,S_{m-n-k,\,k} - S_{m-n-k,\,k}\,S_{m-n-i,\,i} \equiv 0,$$

the $(m-n+1)$ *differential equations* (8) *form a complete system, and the corresponding operations* S *define a group.*[*] The four operations W_1, W_2, O_1, O_2 define a second group, as is well known. There remains to be examined then only the effect of permuting an operator of the first group and one of the second; finding this always expressible as a single operator of the first group, we shall conclude that all these operators together define a group comprehending the others as sub-groups. We find by easy reckoning the following facts:

$$\left\{\begin{aligned}
&S_{m-n,\,0}W_1 - W_1 S_{m-n,\,0} &&= (m-n)\,S_{m-n,\,0},\\
&S_{m-n-1,\,1}W_1 - W_1 S_{m-n-1,\,1} &&= (m-n-1)\,S_{m-n-1,\,1},\\
&\vdots\\
&S_{m-n-r,\,r}W_1 - W_1 S_{m-n-r,\,r} &&= (m-n-r)\,S_{m-n-r,\,r};
\end{aligned}\right.$$

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[*] S. Lie: Theorie der Transformationsgruppen, Erster Abschn., p. 107, Satz 4.

and similarly for W_2,

$$S_{m-n-r,\,r} W_2 - W_2 S_{m-n-r,\,r} = r \,.\, S_{m-n-r,\,r}.$$

The results of permuting operators O with operators S are as follows:

$$\left\{\begin{array}{ll} S_{m-n,\,0}\, O_1 - O_1 S_{m-n,\,0} & = (m-n)\, S_{m-n-1,\,1}, \\ S_{m-n-1,\,1}\, O_1 - O_1 S_{m-n-1,\,1} & = (m-n-1)\, S_{m-n-2,\,2}, \\ \vdots & \\ S_{1,\,m-n-1}\, O_1 - O_1 S_{1,\,m-n-1} & = 1 \,.\, S_{0,\,m-n}, \\ S_{0,\,m-n}\, O_1 - O_1 S_{0,\,m-n} & = 0 \,. \end{array}\right.$$

$$\left\{\begin{array}{ll} S_{m-n,\,0}\, O_2 - O_2 S_{m-n,\,0} & = 0, \\ S_{m-n-1,\,1}\, O_2 - O_2 S_{m-n-1,\,1} & = 1 \,.\, S_{m-n,\,0}, \\ S \vdots & \\ S_{1,\,m-n-1}\, O_2 - O_2 S_{1,\,m-n-1} & = (m-n-1)\, S_{2,\,m-n-2}, \\ S_{0,\,m-n}\, O_2 - O_2 S_{0,\,m-n} & = (m-n)\, S_{1,\,m-n-1}. \end{array}\right.$$

The completeness of the system of equations (6) and (8), thus verified, might have been inferred from the observation that all the operators involved are linear, of the general type known as polar operators, and comprised in an aggregate given by integral indices with definite limits. The above exact expression of the laws of the system shows, however, that as generators of the group, after the operations (6), any single one of the S_{ik} might have been selected. In other words,

The manifoldness of semi-combinants of two binary quantics is but one unit less than that of ordinary invariants.

Beside the extension, now evident, of this theorem to two quantics in any number of variables, there is another not remote. We may consider simultaneous invariants of more than two quantics; e. g. of three binary quantics

$$f_m, \ \phi_n; \ \psi_p \qquad (m \geqq n \geqq p),$$

and define as semi-combinants of the system (in that order) such invariants as are unaltered when these quantics are replaced by the three following:

$$f_m + U_{m-n}\phi_n + V_{m-p}\psi_p, \quad \phi_n + W_{n-p}\psi_p, \quad \psi_p,$$

in which U, V, W denote quantics of the proper orders with arbitrary coefficients. Here we can state the theorem:

The manifoldness of semi-combinants of three binary quantics—whose sequence

must be stated whenever two have the same order in the variables—is less by three units than that of their simultaneous invariants. Only three independent conditions beside those of ordinary invariants are needed to define them completely.

§3. *Semi-Combinant Groundforms defined, with Examples.*

The ternary quantic of order n apolar to a quadric, which is discussed in §1, is a semi-combinant of a given n^{ic} and quadric. It is a linear combination of covariant multiples of the n^{ic} and the quadric. The covariants are all linear in the coefficients of the n^{ic}. We shall characterize this apolar n^{ic} as a *semi-combinant groundform*, and make the following definition:

If two quantics have orders m *and* n *respectively* $(m > n)$, *then any semi-combinant aggregate of covariants of order* m *containing severally one or the other quantic as factors, and linear in the coefficients of the* m^{ic}, *shall be called a semi-combinant groundform of the two quantics.*

Denote two quantics by a_x^m and a_x^n, $(m > n)$. The only covariant linear in the coefficients of a_x^m, of order m, containing a_x^m as a factor is a_x^m multiplied by some invariant of a_x^n. Since other terms of the aggregate are to contain a factor a_x^n, the other factor of each must be a covariant of order $(m - n)$, linear in the coefficients of a_x^m. Since such covariants are to be mentioned often, I denote them uniformly by the initial letter of "linear"—namely, by $L_{m-n}^{(i)}$ or simply $L^{(i)}$. *The general type of a semi-combinant groundform is then*

$$A_x^m = I . a_x^m + (\lambda_1 L' + \lambda_2 L'' + \dots + \lambda_k L^{(k)}) a_x^n, \tag{9}$$

where k *denotes the number of linearly independent covariants* L, I *some invariant of the quantic* a_x^n. That such semi-combinant groundforms exist I will show by a few examples.

For a ternary m^{ic} and 2^{ic} the apolar m^{ic} of Lindemann is, as has been shown (§1), the only semi-combinant groundform. For it was found to be completely determined by the conditions employed, barring a factor which was an arbitrary power of the discriminant of the quadric.

For a binary m^{ic} and 2^{ic} the only semi-combinant groundform is that (conjugate) apolar to the quadric, as I have shown elsewhere.* The same considerations are valid for any number of variables if one of the forms is a quadric.

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* Reduction of the resultant of a binary quadric and n^{ic} by virtue of its semi-combinant property. Bulletin of the American Mathematical Society, Oct. 1894, pp. 11-15.

When a "reduced system" of invariants of two quantics is known, the determination of a semi-combinant groundform is usually easy, since only covariants $L^{(1)}_{m-n}$ enter into the problem, and the condition for a semi-combinant is applied by a single substitution. As first example, consider the binary quartic α_x^4 and cubic $a_x^3 = b_x^3 =$ etc. Following Gordan's method we find four linearly independent covariants of order 1, linear in the coefficients of α_x^4; four covariants L_1:

$$\begin{aligned} L^{\mathrm{I}} &= (a\alpha)^3 \alpha_x, \\ L^{\mathrm{II}} &= (ab)^2(ac)(b\alpha)(c\alpha)^2 \alpha_x, \\ L^{\mathrm{III}} &= (a\alpha)^3(bc)^2(b\alpha)\, c_x, \\ L^{\mathrm{IV}} &= (ab)^2(cd)^2(ae)(b\alpha)(c\alpha)(e\alpha)^2 d_x, \end{aligned}$$

and one invariant of the cubic,

$$I = (ab)^2(cd)^2(ac)(bd).$$

The degrees of these concomitants in the coefficients of a_x^3 are respectively 1, 3, 3, 5; 4. Referring to formula (9) we see that homogeneity admits only the covariants L^{II} and L^{III}, of degree 3 in those coefficients. There are to be determined λ_2 and λ_3 in the expression

$$A_x^4 = I.\alpha_x^4 + (\lambda_2 L^{\mathrm{II}} + \lambda_3 L^{\mathrm{III}})\, d_x^3.$$

In this expression substitute for α_x^4 the conjunctive $\alpha_x^4 + u_x.e_x^3$, and require the terms involving parameters u to disappear identically.

$$\begin{aligned} 0 &= I.d_x^3 u_x + (\tfrac{5}{8}\lambda_2 I.u_x - \tfrac{3}{8}\lambda_3 I.u_x)\, d_x^3, \\ \therefore \quad & 8 + 5\lambda_2 - 3\lambda_3 = 0. \end{aligned} \tag{10}$$

Of this equation there are two linearly independent solutions; as the simplest, we adopt these:

$$\begin{aligned} &1) \quad \lambda_2 = -\tfrac{8}{5}, \; \lambda_3 = 0; \\ &2) \quad \lambda_2 = \quad 0, \; \lambda_3 = \tfrac{8}{3}. \end{aligned}$$

There are thus seen to be two linearly independent semi-combinant groundforms A_x^4, $A_x'^4$, and a simple infinity of combinations $A_x^4 + m.A_x'^4$;

$$A_x^4 = \alpha_x^4 - \tfrac{8}{5}\frac{(ab)^2(ac)(b\alpha)(c\alpha)^2 \alpha_x}{(ab)^2(ac)(bd)(cd)^2} . d_x^3, \tag{11}$$

$$A_x'^4 = \alpha_x^4 + \tfrac{8}{3}\frac{(a\alpha)^3(bc)^2(b\alpha)\, c_x}{(ab)^2(ac)(bd)(cd)^2} . d_x^3. \tag{12}$$

As a second example, let it be required to find all semi-combinant groundforms in the simultaneous system of a ternary quartic and cubic, denoted by α_x^4 and a_x^3 respectively. For the ternary cubic a complete reduced system is given by Gordan.* Our covariants L are to be of order 1, and contain symbols α to the degree 4. The symbolic expression of an L may contain either α_x and three α's in determinant-factors; or an a_x, with four α's in determinant-factors. There correspond to these, in Gordan's tables, mixed concomitants (Zwischenformen) either of class 3, order 0, or of class 4, order 1. Of the former we find two, of the latter three. Their degrees in the coefficients of a_x^3 are respectively 3, 5, 5, 7, 8. There are two invariants, of degrees 4 and 6. With the aid of these there can be formed homogeneous expressions involving the first four covariants L, but none containing that of degree 8.

The symbolic expressions are

1) of the invariants,

$$\begin{aligned} S &= (abc)(abd)(acd)(bcd), \\ T &= (abc)(abd)(ace)(bcf)(def)^2; \end{aligned}$$

2) of the four covariants L,

$$\begin{aligned} L^{\mathrm{I}} &= (abc)(ab\alpha)(ac\alpha)(bc\alpha)\,\alpha_x, \\ L^{\mathrm{II}} &= (abc)(abd)(ace)(bc\alpha)(de\alpha)^2\,\alpha_x, \\ L^{\mathrm{III}} &= (abc)(ab\alpha)(ac\alpha)(bcd)(de\alpha)^2\,e_x, \\ L^{\mathrm{IV}} &= (abc)(abd)(ace)(bcf)(de\alpha)^2(fg\alpha)^2\,g_x. \end{aligned}$$

Instead of determining the most general semi-combinant groundform and singling out four independent ones by particular values given to the parameters, I will for simplicity's sake determine the particular ones, and of them compound the most general.

The four shall be these:

$$\begin{aligned} S.F_1 &= S.\alpha_x^4 + \lambda_1 L^{\mathrm{I}}\;.a_x^3, \\ T.F_2 &= T.\alpha_x^4 + \lambda_2 L^{\mathrm{II}}\,.a_x^3, \\ T.F_3 &= T.\alpha_x^4 + \lambda_3 L^{\mathrm{III}}.a_x^3, \\ S^2.F_4 &= S^2.\alpha_x^4 + \lambda_4 L^{\mathrm{IV}}.a_x^3. \end{aligned}$$

Applying the condition that each of these forms F shall vanish identically for

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* Ueber ternäre Formen dritten Grades, Math. Ann. I, pp. 101–102.

$a_x^4 = a_x^3 . u_x$, I find, after somewhat laborious reduction of the resulting symbolic products,* the values of the constants

$$\lambda_1 = -2,\ \lambda_2 = -2,\ \lambda_3 = -6,\ \lambda_4 = \qquad ;$$

so that the simple semi-combinants $F_1 \ldots . F_4$ have the expressions

$$\left.\begin{aligned} F_1 &= a_x^4 - 2\frac{L^{\mathrm{I}}}{S} . a_x^3, \\ F_2 &= a_x^4 - 2\frac{L^{\mathrm{II}}}{T} . a_x^3, \\ F_3 &= a_x^4 - 6\frac{L^{\mathrm{III}}}{T} . a_x^3, \\ F_4 &= a_x^4 - 36\frac{L^{\mathrm{IV}}}{S^2} . a_x^3. \end{aligned}\right\} \tag{13}$$

The most general semi-combinant groundform is the linear combination of these four, with arbitrary multipliers,

$$F = l_1 F_1 + l_2 F_2 + l_3 F_3 + l_4 F_4 .$$

From these examples it is sufficiently clear what method must be followed in obtaining all semi-combinant groundforms of the system of two given quantics. The covariants L must first be enumerated. While not every L will give necessarily an independent semi-combinant, yet none gives more than one; and we may formulate the following conclusion:

The number of linearly independent semi-combinant groundforms of two quantics of different order does not exceed the number of linearly independent covariants L, linear in the coefficients of the higher quantic, and of order equal to the difference in orders of the given quantics.

Thus a superior limit is established, which will be confirmed later, when also it will appear why this number falls often below that limit.

§4. *Reduced Form-system of Semi-Combinants derivable from that of General Covariants.*

Any covariant of a system of two quantics a_x^m, a_x^n, which is also a semi-combinant of the system, is unaltered when in its explicit formula the coefficients of

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* The most useful reduction-formulæ are given by Clebsch and Gordan, Ueber cubische ternäre Formen, Math. Ann. 6, pp. 448-9 and 467.

a_x^m are replaced by those of $a_x^m + u_x^{m-n} a_x^n$. This is indeed the definition of a semi-combinant. Accordingly, giving the arbitrary parameters in u_x^{m-n} special values, so that $a_x^m + u_x^{m-n} a_x^n$ becomes a semi-combinant groundform A_x^m, we may notice in particular that

All semi-combinant invariants and covariants are unaltered by the substitution

$$a_x^m \sim A_x^m,$$

A_x^m denoting any semi-combinant groundform.

Assume that in every system such semi-combinant groundforms exist. Suppose the aggregate of covariants of the system subjected to the above substitution,

$$a_x^m \sim A_x^m.$$

Every covariant will be transformed into a semi-combinant, for it becomes a covariant of a semi-combinant. Suppose further all identical equations to be explicitly given, which express covariants in terms of the finite number of covariants constituting a reduced form-system. These syzygies being subjected to the same substitution,

$$a_x^m \sim A_x^m,$$

will become syzygies between semi-combinants, reducible covariants becoming reducible semi-combinants, and the reduced form-system of ordinary covariants yielding by transformation a set of semi-combinants in terms of which all other semi-combinants are expressed as rational entire functions. Hence the theorem:

The reduced form-system of semi-combinants cannot contain more irreducible concomitants than that of ordinary covariants.

That it must contain fewer will appear subsequently (see §6); for at least one covariant of the reduced system will vanish identically.

This transformation gives all the semi-combinants of a system, and a part of their relations *inter se*, as soon as all covariants of the system, their syzygies, and a single semi-combinant groundform are known. The discovery of the latter is therefore the essential step in proceeding from a theory of ordinary invariants, such as already exists in greater or less completeness, to a correspondingly complete theory of the subgroup, semi-combinants. Particular examples like those of §3 do not suffice. We must address ourselves to the general question, how to find semi-combinant groundforms in the system of two or more given quantics.

§5. *Affiliated Forms defined and determined as Covariants in a given System.*

A class of concomitants that are certainly semi-combinant groundforms of two given quantics a_x^m and α_x^n, $(m > n)$, are those satisfying identically linear differential equations, invariant *per se*, having as coefficients functions of the coefficients of the quantic α_x^n; provided such solutions are completely determinate. (Otherwise not the particular solution, but the aggregate of all solutions, would be semi-combinant.) The defining equations must equal in number the arbitrary parameters in the conjunctive $a_x^m + u_x^{m-n}\alpha_x^n$. They may comprise that number of invariants, each equated to zero, or any part of the number may be represented by a covariant equated identically to zero—an *invariantive set* of differential equations. The solutions will be rational covariants if the defining equations are all linear; were any of higher degree the solutions would still be semi-combinants, but irrational. I intend here to examine solutions of sets of equations found by requiring a covariant of order $m-n$ to vanish identically. I shall speak of these equations in the aggregate as a *characteristic equation*, and call their solution an *affiliated form* or *affiliant* of the system of a_x^m and α_x^n.

An affiliated form or affiliant of the system of a_x^m *and* α_x^n *is any covariant of the type* $A_x^m = a_x^m + l_x^{m-n}\alpha_x^n$, *which satisfies a characteristic equation obtained by equating identically to zero a covariant of order* $m-n$, *linear in the coefficients of* A_x^m.

If the quantic of lower order be a quadric, there is of course only one affiliated form,* that one, namely, which is apolar or conjugate to the quadric. Two examples of this simple case will make the above definition more intelligible.

First example: System of binary cubic and quadric.

Required the values of l_1 and l_2 which render $a_x^3 + l_x . \alpha_x^2 = A_x^3$ an affiliated form. There are two covariants of A_x^3 and α_x^2 which can give equations of definition,† but the identical vanishing of either one involves the vanishing of the other. The one of lowest degree is this:

$$(\alpha A)^2 A_x \equiv 0, \qquad (14)$$

or expanding,

$$3\,(a\alpha)^2 a_x + (\alpha\beta)^2 l_x + 2\,(\alpha\beta)(\alpha l)\,\beta_x \equiv 0,$$

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* For the condition of apolarity involves every other covariant condition of the prescribed order.

† Named p and q by Gordan-Kerschensteiner, p. 328.

and reducing,

$$3\,(a\alpha)^2\alpha_x + 2\,(ab)^2 l_x \equiv 0.$$

Since the only term involving the parameters l_1, l_2 contains the factor l_x, it is only necessary to solve for this linear form

$$l_x = -\tfrac{3}{2}\,.\,\frac{(a\alpha)^2\,\alpha_x}{(ab)^2}\,; \tag{15}$$

hence the required affiliated form is

$$A_x^3 = \alpha_x^3 - \tfrac{3}{2}\,.\,\frac{(a\alpha)^2\,\alpha_x\,.\,b_x^2}{(ab)^2}\,. \tag{15a}$$

Second example: System of binary quartic and quadric.

Of the six quadratic covariants of a quartic and a quadric, two are linear in the coefficients of the quartic, those named ψ and Ψ by Clebsch (*Binäre Formen*, pp. 213, 214). If ψ become identically zero, so also will Ψ. The two give thus only one characteristic equation for an affiliated form: $A_x^4 = \alpha_x^4 + l_x^2 . \alpha_x^2$, namely,

$$(aA)^2 A_x^2 \equiv 0, \tag{16}$$

i. e. $$6\,(a\alpha)^2\,\alpha_x^2 + (ab)^2\,l_x^2 + 4\,(ab)(al)\,b_x\,l_x + (al)^2\,b_x^2 \equiv 0\,;$$

or, after reducing one term,

$$6\,(a\alpha)^2\,\alpha_x^2 + 3\,(ab)^2\,l_x^2 + (al)^2\,b_x^2 \equiv 0\,.$$

This is equivalent to three equations for parameters, and these are linearly independent, for otherwise we find that a multiple of the discriminant of the quadric must vanish.

Instead of solving these three equations, compounding the resultant determinant-quotients with proper multipliers into the form l_x^2, and reducing its expression to covariant form, I find it more convenient to solve by a convergent series. The equation for l_x^2 is, calling $(ab)^2 = \Delta$,

$$l_x^2 = -\,2\,\frac{(a\alpha)^2\,\alpha_x^2}{\Delta} - \tfrac{1}{3}\,b_x^2\,.\,\frac{(al)^2}{\Delta}\,.$$

By repeated substitution of this expression for l_x^2 in the second member—in itself—we have

$$l_x^2 = -\,\frac{2\,(a\alpha)^2\,\alpha_x^2}{\Delta} + \tfrac{2}{3}\,\frac{(a\alpha)^2(b\alpha)^2\,c_x^2}{\Delta^2} - \tfrac{2}{9}\,\frac{\Delta.(a\alpha)^2(b\alpha)^2\,c_x^2}{\Delta^3} + \text{etc.},$$

$$l_x^2 = -\,2\,\frac{(a\alpha)^2\,\alpha_x^2}{\Delta} + \frac{(a\alpha)^2(b\alpha)^2\,c_x^2}{\Delta^2}\left(\tfrac{2}{3} - \tfrac{2}{9} + \tfrac{2}{27} - \ldots\right)$$

$$= -\,2\,\frac{(a\alpha)^2\,\alpha_x^2}{\Delta} + \tfrac{1}{2}\,.\,\frac{(a\alpha)^2(b\alpha)^2\,c_x^2}{\Delta^2}. \tag{17}$$

Therefore the affiliated quartic of the system is

$$A_x^4 = \alpha_x^4 - 2\,\frac{(a\alpha)^2\,\alpha_x^2}{\Delta}\,.\,b_x^2 + \tfrac{1}{2}\,.\,\frac{(a\alpha)^2.(b\alpha)^2}{\Delta^2}\,.\,c_x^2\,.\,d_x^2\,. \tag{17a}$$

These examples exhibit one mode of finding the affiliated form when its characteristic equation is given. Such a solution must be a semi-combinant groundform, for it is a covariant by virtue of the covariant defining equation; it is of the type $\alpha_x^m + l_x^{m-n} a_x^n$; and as it is uniquely determined from the conditions, it must be unaltered by the substitution

$$\alpha_x^m \sim \alpha_x^m + u_x^{m-n} a_x^n,$$

for that affects neither its type nor its characteristic equation.

This last remark may be stated more clearly by employing a functional notation for covariants, and by this means an important property of the covariant l_x^{m-n} or L may be developed. Designate by α and a the quantics α_x^m and a_x^n; by U any arbitrary quantic u_x^{m-n}; by $G(\alpha)$ and $L(\alpha)$ two covariants, of order $m-n$, of α_x^m and a_x^n, linear in the coefficients of the former, such that $L(\alpha)$ satisfies the characteristic equation

$$G(\alpha + aL(\alpha)) \equiv 0. \tag{18}$$

On account of the linear structure of these covariants we have

$$\left.\begin{aligned} G(\alpha + aL(\alpha)) &= G(\alpha) + G(a.L(\alpha)),\\ L(\alpha + a.U) &= L(\alpha) + L(aU). \end{aligned}\right\} \tag{19}$$

If for α_x^m we substitute $\alpha_x^m + u_x^{m-n} a_x^n$, we have from the identical equation (18),

$$G(\alpha + aU + aL(\alpha + aU)) \equiv 0\,;$$

and therefore, since L is uniquely determined by (18), the identity

$$\alpha + aL(\alpha) \equiv (\alpha + aU) + aL(\alpha + aU), \tag{20}$$

showing that the affiliant $\alpha + aL(\alpha)$ is unaltered by the substitution $\alpha_x^m \sim \alpha_x^m + u_x^{m-n} a_x^n$; or, in other words, *every affiliant is a semi-combinant groundform of the system.*

Equation (20) expanded by (19) is this:

$$\alpha + aL(\alpha) \equiv \alpha + aU + aL(\alpha) + aL(a.U),$$

or, on dropping like terms,

$$aU + aL(aU) \equiv 0,$$

and without the factor a,

$$L(aU) \equiv -\,U. \tag{21}$$

Remembering that L is determined as the quotient of an integral rational simultaneous covariant of α and a by an invariant of a alone, we may set

$$L(\alpha) = \frac{K(a, \alpha)}{I(a)},$$

whereby equation (21) becomes

$$K(a, aU) + U.I(a) \equiv 0. \tag{21a}$$

This may be regarded as a defining equation for $K(a, \alpha)$, in accordance with which it is to be derived, linear in the coefficients of α, from a suitable given invariant $I(a)$. The solution of this problem would lead directly to the formation of a semi-combinant groundform.

In a given system, how many affiliants can occur? Certainly not more than the number of independent characteristic equations; that is, than the number of independent covariants G, G', G'', ... of order $m - n$, linear in the coefficients of the higher quantic. Hence, as was seen above in the closing theorem of §3, *the number of independent affiliants has for an upper limit the limit of the number of semi-combinant groundforms in the same system.*

Further, there is an obvious reason why not all linearly independent covariants G, G', G'', etc., need furnish different affiliants when they are employed in characteristic equations. It may happen that one or more of the remaining G's can be represented as a simultaneous covariant of the first and of a_x^n. In that case, of course, the solution of $G = 0$ would satisfy $G' = 0$ and the rest belonging to the system of G and a. *This case always occurs* if the characteristic equation has a determinate solution whose covariant L is not a mere multiple of G.

If we suppose, as in (18),

$$G\big(\alpha + aL(\alpha)\big) = 0,$$

then we have, according to (21),

$$L(aU) = -U, \text{ and } L\big(aL(\alpha)\big) = -L(\alpha).$$

Therefore the solution L_1 of the equation

$$L\big(\alpha + aL_1(\alpha)\big) \equiv 0$$

is L itself. For we have

$$L\big(\alpha + aL(\alpha)\big) \equiv L(\alpha) + L\big(aL(\alpha)\big) \equiv L(\alpha) - L(\alpha) \equiv 0, \tag{22}$$

and this we will adopt as the normal form of the characteristic equation of an affiliant.

Here, of course, $L(\alpha)$ may be radically different from $G(\alpha)$, or it may differ only by a factor. In either case, $L(\alpha)$ must be a covariant of $G(\alpha)$; for by virtue of (18) and (22), if $L(A) \equiv 0$, it follows that $G(A) \equiv 0$, and conversely.

We can name not only a superior limit, but probably also one inferior limit to this number of independent affiliants. *This number is not less than the number of irreducible invariants of the lower quantic a_x^n, diminished by the number of syzygies of the first kind.* From such invariants there can be derived covariants $G(\alpha)$ by an Aronhold process, which shall replace coefficients of a_x^n by those of the polar, $a_x^s a_y^{n-s}$. The latter will be independent save as derivatives of the syzygies shall indicate relations. Such covariants G will not vanish simultaneously except by virtue of special invariant properties of the lower quantic. There is lacking a proof that each such G has corresponding to it a determinate solution L of the equation

$$G(\alpha + aL(\alpha)) \equiv 0.$$

As I have no complete proof of this, it may be left for the present to be investigated in each separate case discussed.*

A question of some interest may be mentioned here. Suppose two affiliants determined by characteristic equations, and the latter put into normal form (*vid.* (22)), so that we may write

$$L_1(\alpha + aL_1(\alpha)) \equiv 0,$$
$$L_2(\alpha + aL_2(\alpha)) \equiv 0.$$

If now these equations be combined linearly, how will the solution depend upon the solutions L_1 and L_2 of the separate equations? If, namely,

$$G_3(\alpha + aL_3(\alpha)) = (\lambda_1 . L_1 + \lambda_2 . L_2)(\alpha + aL_3(\alpha)) \equiv 0, \qquad (23)$$

what relation subsists between L_1, L_2, L_3?

Expanding (23) by (19), and using (21), we find

$$\lambda_1 . L_1(\alpha) + \lambda_2 . L_2(\alpha) - \lambda_1 . L_3(\alpha) - \lambda_2 . L_3(\alpha) \equiv 0,$$
$$L_3(\alpha) = \frac{\lambda_1 . L_1(\alpha) + \lambda_2 . L_2(\alpha)}{\lambda_1 + \lambda_2}. \qquad (24)$$

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* The most obvious point of attack for this problem is offered by equation (21a).

The same considerations show that for any number of equations, $G_i(\alpha + aL_i) \equiv 0$, with determinate solutions L_i,

$$\Sigma\lambda_i L_i\left(\alpha + \frac{a.\Sigma\lambda_i L_i(\alpha)}{\Sigma\lambda_i}\right) \equiv 0. \tag{24a}$$

Loosely stated, this would show that the solution of $\Sigma\lambda_i L_i \equiv 0$ is $\frac{\Sigma\lambda_i L_i}{\Sigma\lambda_i}$.

The linear aggregate of characteristic equations is projectively related to the aggregate of corresponding solutions, and is perspectively related to them, if the characteristic equations are in normal form.

For the solution of (24a) is identically

$$\frac{\Sigma\lambda_i.\alpha + a\Sigma\lambda_i L_i(\alpha)}{\Sigma\lambda_i},$$

and this is

$$\frac{\Sigma\lambda_i\big(\alpha + aL_i(\alpha)\big)}{\Sigma\lambda_i},$$

wherein every term of the numerator $\big(\alpha + aL_i(\alpha)\big)$ is the solution of the corresponding characteristic equation

$$L_i \equiv 0.$$

This shows how a given equation may have its solution indeterminate. If in (24a) we assume $\Sigma\lambda_i = 0$, the solution would become

$$\Sigma\lambda_i\alpha + a\Sigma\lambda_i L_i(\alpha) = a.\Sigma\lambda_i L_i(\alpha).$$

Since this is necessarily a semi-combinant, the solution of characteristic equation

$$\Sigma\lambda_i L_i(\alpha + aU) \equiv 0$$

is indeterminate. It is namely the quantic a_x^n multiplied by an arbitrary quantic U of order $m - n$; for we have by (21)

$$\Sigma\lambda_i L_i(aU) \equiv -\Sigma\lambda_i.U \equiv U.\Sigma\lambda_i \equiv 0.$$

Equations of this sort may be regarded as forming a singular system for defining the quantic a_x^n, which is indeed a semi-combinant by virtue of being a covariant and unaltered by the substitution

$$a_x^m \sim a_x^m + u_x^{m-n}.a_x^n.$$

Every linear combination of normal characteristic equations has a determinate solution except when the combination $\Sigma\lambda_i L_i \equiv 0$ is one of the singular system having $\Sigma\lambda_i = 0$. Of any singular characteristic equation the solution is indeterminate, being the quantic of lower order multiplied by an arbitrary function.

§6. *Every Semi-Combinant Groundform is an Affiliant. Production of its Characteristic Equation.*

The inquiry for a single semi-combinant groundform in the system of two quantics α_x^m, a_x^n (or α and a) resulted in the finding of a whole class of such groundforms, the affiliants of §5. Every affiliant is a semi-combinant groundform, and as such suffices for the determination of all covariants that share its semi-combinant property. It is now to be shown that, conversely, every semi-combinant groundform is an affiliant.

Let the rational covariant of α_x^m and a_x^n:

$$\left.\begin{array}{ll} & A_x^m \equiv \alpha_x^m + l_x^{m-n} a_x^n, \\ \text{or for brevity} & A \equiv \alpha + a.L(\alpha), \end{array}\right\} \qquad (25)$$

be a semi-combinant; i. e. let it be unaltered by the substitution

$$\alpha_x^m \sim \alpha_x^m + u_x^{m-n} a_x^n.$$

Then the N_m equations involved in the identity or definition (25) may be regarded as equations of the first degree in coefficients of α, by whose aid those coefficients are to be expressed as rational linear functions of coefficients of A. But the quantic α is by hypothesis not determinate when A is known; on the contrary, it contains implicitly the N_{m-n} parameters of u_x^{m-n}. Therefore all determinants of a matrix having N_m rows and $(N_m - N_{m-n})$ columns—*linear* in coefficients of A, but not usually so in those of α—must vanish. These several zero-determinants can be multiplied with suitable power products of variables $(x_1, x_2, x_3, \ldots . x_r)$, so that their sum equated to zero will constitute the covariant characteristic equation described in §5, by virtue of which the quantic A is an affiliant. This process of proof is, however, unnecessarily expanded, and can be compressed into the following symbolic form, a reproduction of (22).

Since A is a semi-combinant,

$$\alpha + a.U + aL(\alpha + aU) \equiv \alpha + aL(\alpha);$$

therefore

$$aU + aL(aU) \equiv 0,$$

$$L(aU) \equiv - U,$$

U denoting an arbitrary quantic $U = u_x^{m-n}$. This property of the covariant $L(\alpha)$

depends only upon the order $m-n$ of the quantic U. Hence for U we may write L itself:

$$L(aL(\alpha)) \equiv -L(\alpha),$$

or

$$L(\alpha)+L(aL(\alpha)) \equiv L(\alpha+aL(\alpha)) \equiv 0;$$

and this is exactly the characteristic equation of an affiliant in its normal form (*vid.* (22)). This theorem may be formulated as follows:

If a semi-combinant groundform be written in symbols of two quantics α and a

$$A=\alpha+a.L(\alpha),$$

where** $L(\alpha)$ **is a rational covariant, linear in the coefficients of** α, **then this semi-combinant groundform is an affiliant of** a, **satisfying the equation

$$L(A) \equiv 0.$$

As L is always fractional, it remains only to multiply it by a suitable invariant of a to make it an entire covariant of A and a.

§7. *Further Examples of Affiliants.*

In accordance with the foregoing, we can write immediately the characteristic equations of the semi-combinants calculated in §3. Of the binary quartic and cubic, the semi-combinant A_x^4 satisfies the defining equation (see (11))

$$(ab)^2(ac)(bA)(cA)^2A_x \equiv 0.$$

This is obviously derived from the discriminant of the cubic

$$(ab)^2(ac)(bd)(cd)^2.$$

The other semi-combinant groundform $A_x'^4$ (*vid.* (12)) has for its equation

$$(aA)^3(bc)^2(bA)\,c_x \equiv 0.$$

This can be simplified by equating to zero its transvectant with the Hessian of the cubic: $(de)^2 d_x e_x$. This gives

$$(aA)^3(bc)^2(bA)(cd)(de)^2 e_x \equiv 0.$$

One reduction brings this into the form

$$(aA)^3 A_x.(bc)^2(be)(cd)(de)^2 \equiv 0,$$
$$\therefore\ (aA)^3 A_x \equiv 0,$$

if the cubic a_x^3 has no square factor. This is evidently a derivative of the identically vanishing invariant $(ab)^3$. Just as the two linear covariants $L'''_{(a)}$ and $L'_{(a)}$ are found to give characteristic equations having the same solution $A' \; (= A_x^{\prime 4})$,

$$L'(A') \equiv 0, \quad L'''(A') \equiv 0,$$

so the solution $A \, (= A_x^4$, formula (11)) of $L''(A) \equiv 0$ satisfies the fourth, the only remaining independent linear covariant equation $L^{\text{IV}}(A) \equiv 0$. For L^{IV} is the transvectant of L'' with the Hessian of the cubic.

In the second example, the system of the ternary quartic and cubic, the characteristic equations of F_1, F_2, F_3, F_4 can be written down directly from the symbolic expressions of L^{I}, L^{II}, L^{III}, L^{IV}. Our special interest will be directed toward the fifth linear covariant $L^{\text{V}}(\alpha)$ of degree 8 in the coefficients of the cubic. By reason of its degree alone it was found impossible to have a corresponding semi-combinant groundform, since there is no invariant of degree 9. But if we use this covariant in a characteristic equation for an affiliant, must it not disclose a corresponding semi-combinant? Making trial, I learn that $L^{\text{V}}(\alpha)$ *is itself a semi-combinant, and hence cannot determine an affiliant.* The proof is condensed by the use of Gordan's notation and reductions as follows:

$$L^{\text{V}}(\alpha) = \alpha_t^2 \alpha_s^2 (stx), \tag{26}$$

where u_t^3 and u_s^3 are fundamental contravariants,

$$\left.\begin{aligned} u_s^3 &= (abc)(abu)(acu)(bcu), \\ u_t^3 &= a_s b_s u_s \, (abu)^2. \end{aligned}\right\} \tag{26a}$$

To prove $L^{\text{V}}(\alpha)$ a semi-combinant we must show that

$$L^{\text{V}}(\alpha + Ua) \equiv L^{\text{V}}(\alpha); \text{ i. e. } L^{\text{V}}(Ua) \equiv 0.$$

Since U and a denote u_x and a_x^3, we have

$$\begin{aligned} 2L^{\text{V}}(Ua) &= u_t a_t a_s^2 (stx) + a_t^2 a_s u_s (stx) \\ &= \tfrac{1}{3} S.(ttx)\, u_t{}^{*} + \tfrac{1}{3} T.(xss)\, u_s{}^{\dagger} \equiv 0, \end{aligned}$$

both parts vanishing identically.

When the two fundamental quantics are of orders differing by one, and the lower order is even, one affiliant can be found very simply, whatever the number

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[*] Gordan, Ueber ternäre Formen dritten Grades, Math. Ann. I, p. 104, Tafel IV, 2.

[†] Ibidem, p. 108, Tafel XII, 5. Cf. Math. Ann. VI, p. 467.

of variables in the two quantics. Assuming two quaternary quantics of orders $2n$ and $2n+1$,

$$\alpha_x^{2n+1} \text{ and } a_x^{2n},$$

I mean the affiliant $\alpha_x^{2n+1} + l_x . a_x^{2n} \equiv A_x^{2n+1}$,

which satisfies the characteristic equation

$$(abcA)^{2n} A_x \equiv 0. \tag{27}$$

Expanding this equation,

$$\begin{aligned} 0 &\equiv (abc\alpha)^{2n}\alpha_x + \frac{1}{2n+1}(abcd)^{2n} l_x + \frac{2n}{2n+1}(abcd)^{2n-1}(abcl)\, d_x \\ &\equiv (abc\alpha)^{2n}\alpha_x + \frac{n+2}{4n+2}(abcd)^{2n} l_x; \\ \therefore\ l_x &= -\frac{4n+2}{n+2} \cdot \frac{(abc\alpha)^{2n}\alpha_x}{(abcd)^{2n}}, \\ A_x^{2n+1} &= \alpha_x^{2n+1} - \frac{4n+2}{n+2} \cdot \frac{(abc\alpha)^{2n}\alpha_x}{(abcd)^{2n}} . d_x^{2n}. \end{aligned}$$

For quantics in r variables, the same equation will give

$$A_x^{2n+1} = \alpha_x^{2n+1} - \frac{r(2n+1)}{2n+r} \cdot \frac{(ab \ldots . \alpha)^{2n}\alpha_x}{(ab \ldots . d)^{2n}} . d_x^{2n}. \tag{28}$$

Two binary quantics, the lower of odd order, a_x^{2n+1}, the other of order higher by unity, α_x^{2n+2}, have an affiliant determined almost as simply as the above. Set $A_x^{2n+2} = \alpha_x^{2n+2} + l_x a_x^{2n+1}$, and take as characteristic equation

$$(aA)^{2n+1} A_x \equiv 0. \tag{29}$$

This is, when expanded,

$$(a\alpha)^{2n+1}\alpha_x + \frac{1}{2n+2}(ab)^{2n+1} l_x + \frac{2n+1}{2n+2}(ab)^{2n}(al)\, b_x \equiv 0.$$

Since the second term contains the vanishing invariant $(ab)^{2n+1}$, the equation becomes

$$(a\alpha)^{2n+1}\alpha_x + \frac{2n+1}{2n+2}(ab)^{2n}(al)\, b_x \equiv 0.$$

For convenience, denote by p_x the left-hand member of this equation, and by k_x^2

the covariant $(cd)^{2n}c_x d_x$. Since $p_x \equiv 0$, we may equate to zero its transvectant,

$$0 \equiv (pk)\,k_x \equiv (cd)^2\ (a\alpha)^{2n+1}(\alpha c)\,dx + \frac{2n+1}{2n+2}\,(ab)^{2n}(cd)^{2n}(al)(bd)\,c_x$$

$$\equiv (cd)^{2n}(a\alpha)^{2n+1}(\alpha c)\,d_x + \frac{2n+1}{4n+4}\,(ab)^{2n}(cd)^{2n}(ac)(bd)\,.\,l_x,$$

$$l_x = +\,\frac{4\,(n+1)}{2n+1}\,.\,\frac{(ab)^{2n}(c\alpha)^{2n+1}(a\alpha)\,b_x}{(ab)^{2n}(cd)^{2n}(ac)(bd)}\,,$$

$$A_x^{2n+2} = \alpha_x^{2n+2} + \frac{4\,(n+1)}{2n+1}\,.\,\frac{(ab)^{2n}(c\alpha)^{2n+1}(a\alpha)\,b_x}{(ab)^{2n}(cd)^{2n}(ac)(bd)}\,.\,d_x^{2n+1}. \tag{30}$$

Whatever the number of variables, if the orders of the two quantics differ only by unity, the *discriminant* of the lower quantic furnishes by evectation a characteristic equation whose solution can be obtained immediately as in (27) and (28). *Thus there is proved in general the existence of an affiliant when the given forms are of order* n *and* $n+1$. Where the difference in orders is two or more, the existence of affiliants is yet to be confirmed.

As showing interesting processes capable of extension, consider a binary quartic and sextic, a_x^4 and α_x^6. Let it be required to determine the affiliant

$$A_x^6 = \alpha_x^6 + l_x^2 a_x^4$$

from the equation
$$(aA)^4 A_x^2 \equiv 0,$$

or l_x^2 from the reduced equation

$$(a\alpha)^4\alpha_x^2 + \tfrac{1}{3}\,(ab)^4 l_x^2 + \tfrac{2}{5}\,(ab)^2(al)^2\,b_x^2 \equiv 0.$$

Transposing, this gives

$$l_x^2 = -\,3\,\frac{(a\alpha)^4\alpha_x^2}{(ab)^4} - \tfrac{6}{5}\,\frac{(la)^2(ab)^2\,b_x^2}{(ab)^4}\,. \tag{31}$$

The scheme for solving this equation without leaving the domain of rational covariants is now to substitute the value of l_x^2, given in this second member, into the second term of that second member, then to repeat the process *ad infinitum*. This is admissible if the resulting series shall be found to converge. For the convenient execution of this plan I will designate by P the operation which from any quadric u_x^2 derives $(ua)^2 a_x^2$. Then for the second term of the second member of (31) I have

$$-\,\frac{6}{5}\,\frac{(la)^2(ab)^2\,b_x^2}{(ab)^4} = r\,.\,P^2\,(l_x^2),$$

where
$$i = (ab)^4,\quad r = -\,\frac{6}{5i}\,.$$

Calling the first term F,

$$F = -\frac{3}{i}(a\alpha)^4 \alpha_x^2,$$

equation (31) gives the series

$$\begin{aligned} l_x^2 &= F + rP^2(F) + r^2P^4(F) + r^3P^6(F) + \ldots. \\ &= (1 + rP^2 + r^2P^4 + r^3P^6 + \ldots.)(F). \end{aligned} \tag{32}$$

The problem is reduced to the summation of the infinite series of operations upon a quadric F,

$$C(F) = (1 + rP^2 + r^2P^4 + \ldots.)(F). \tag{33}$$

There is a reduction-formula given by Clebsch* which is, when adapted for even powers of the operator P: $(j = (ab)^2(bc)^2(ca)^2)$,

$$P^k = i.P^{k-2} - \frac{i^2}{4}P^{k-4} + \frac{j^2}{9}P^{k-6}.$$

Applying this in the series C after the third term, I have

$$C\left(1 + \frac{6}{5} + \frac{9}{25}\frac{24j^2}{125i^3}\right) = \left(1 + \frac{6}{5} + \frac{9}{25}\right) - P^2\left(\frac{6}{5i} + \frac{36}{25i}\right) + \frac{36}{25i^2}P^4.$$

Since P^4 can be reduced,

$$P^4 = \frac{i}{2}P^2 + \frac{j}{3}P,$$

this gives the sum,

$$C = \frac{40i^3}{40i^3 + 3j^2} + \frac{15ij}{80i^3 + 6j^2}P - \frac{30i^2}{40i^3 + 3j^2}P^2.$$

Substituting this in (33) and (32), together with the value of F, there results

$$l_x^2 = -\frac{15}{40i^3 + 3j^2}\{8i^2 + \tfrac{3}{2}jP - 6iP^2\}((a\alpha)^4\alpha_x^2),$$

or

$$l_x^2 = -\frac{15(a\alpha)^4}{40i^3 + 3j^2}\left(8i^2.\alpha_x^2 + \tfrac{3}{2}j.(b\alpha)^2b_x^2 - 6i.(b\alpha)^2(bc)^2c_x^2\right). \tag{35}$$

This completes the determination of the affiliant

$$A_x^6 = a_x^6 + l_x^2.a_x^4.$$

A somewhat simpler affiliant exists, a linear combination of the one just found with that whose characteristic equation is derived from the invariant j,

$$(ab)^2(aA')^2(bA')^2A_x'^2 \equiv 0.$$

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*Clebsch, Binäre Formen, p. 219, (2).

The simplest affiliant should be used in discussing the reduced form-system of semi-combinants. For that purpose, however, the simplest would probably be *not that whose symbolic expression is of simplest type, but that whose characteristic equation is of simplest type.*

This last example gives a form of solution which, if applicable in any particular case, is certainly more elegant than the separation of the characteristic equation into its constituent parts, the determination of the auxiliary quantic l_x^{m-n} from these parts as a determinant quotient, and the reduction of this quotient to simpler covariants. The longer process, however, gives occasion to investigate directly whether the solution is determinate. For any one case such an investigation can be made to depend upon a canonical form of the quantic of lower order, if a canonical form is known. An interesting alternative is offered by the series $C(F)$, formula (33). In general, P would be replaced by some covariant process which preserves (as P does) the order of the operand in the variables and its degree in the coefficients, say by the process Π. The generalized problem would be to find under what conditions the series

$$(1 + r\Pi + r^2\Pi^2 + \ldots . \text{ in inf.})(F)$$

represents a uniformly convergent series, where r denotes some invariant factor.

Only one such series fully treated has met my eye, that referred to above in Clebsch's treatment of the system of a binary quartic and quadric. By the aid of his results it is possible to find affiliants of any order n to the binary quartic a_x^4, which shall satisfy the equation

$$(aA)^4 A_x^{n-4} \equiv 0,$$

or the equation

$$(ab)^2 (aA)^2 (bA)^2 A_x^{n-4} \equiv 0.$$

I should mention, of course, the system of a quadric and an n^{ic}, where the summation of a geometric series is sufficient for calculating an affiliant or apolar quantic.

§8. *Applicability to Normal-form Problem, with a Special Theorem.*

In geometric research, two quantics equated to zero denote two algebraic loci. The relative positions of these two loci may be investigated projectively, when the aggregate of all simultaneous covariants must be discussed. If, however, not the two loci but their intersection is to be investigated projectively, then the sub-group, their semi-combinants, is alone to be discussed. For the intersection of two loci,

$$a_x^m = 0,\ a_x^n = 0, \qquad (m > n),$$

is equally the intersection of the loci

$$a_x^m + u_x^{m-n} a_x^n = 0,\ a_x^n = 0\,;$$

that is, of the lower-order locus with the conjunctive of the two. Projective covariants of the intersection-locus are therefore covariants of the conjunctive and of the lower-order locus, independent of the arbitrary coefficients of u_x^{m-n}; that is to say, they are semi-combinants of a_x^m and a_x^n. Ordinarily speaking, an intersection-locus is a geometric form when the two quantics contain at least three variables. Thus semi-combinants of two ternary quantics belong peculiarly to the complete system of points in which two algebraic curves intersect. More interesting is the consideration that projective properties of such twisted curves as are complete intersections of two algebraic surfaces ("elementary curves") will be represented by relations among semi-combinants of the intersecting surfaces (i. e. of their associated quantics).* For point-systems in three dimensions, or for curves in more than three dimensions, systems of at least three quantics and their semi-combinants will enter the discussion. If, further, the orders of two intersecting loci be equal, then not semi-combinants but complete combinants (doubly semi-combinants) of these two quantics must be considered.

A special problem arises in the theory of Abelian integrals on "elementary" twisted curves, i. e. on the non-singular intersection-curve of two algebraic surfaces, usually of unequal order. Normal-forms for integrals of the different species are to be fixed by algebraic considerations. For the reasons above specified, a normal-form ought obviously to be a semi-combinant of the two surfaces. For this as well as for other applications, the following theorem is valuable:

Theorem: If a covariant of two quantics of unlike order, $F(a_x^m, a_x^n)$ *or* $F(a, a)$ *has been found which changes only by multiples of* a_x^m *and* a_x^n *when subjected to the substitution*

$$a_x^m \sim a_x^m + u_x^{m-n} a_x^n,$$

or

$$a \sim a + Ua,$$

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* Semi-combinants of the two surfaces would be, for example, the tact-invariant, the equation of the developable having the intersection-curve for its edge of regression, and the equation of the locus of triple-secants of the intersection-curve.

then a semi-combinant of α *and* a *can be derived from* $F(\alpha, a)$ *by the addition of covariant multiples of* α_x^m *and* a_x^n,

$$F'(\alpha, a) = F'(\alpha + U a, a) = F(\alpha, a) + M.\alpha + N.a,$$

where M *and* N *denote covariants.*

In proof of this I assume, what is not yet fully demonstrated, that there exists an affiliant groundform $A = \alpha + L.a$. In use, this must be tested until it shall be generally proven. Substituting A for α gives a semi-combinant

$$F'(\alpha, a) = F(A, a) = F(\alpha + a.L, a).$$

But by hypothesis

$$F(\alpha + a.L, a) = F(\alpha, a) + \text{multiples of } \alpha \text{ and } a,$$

(say) $$= F(\alpha, a) + M.\alpha + N.a;$$

that is to say, *the semi-combinant* $F'(\alpha, a)$ *differs from* $F(\alpha, a)$ *by covariant multiples of* α *and* a, *q. e. d.*

For precision in any proposed application the assumption spoken of must be verified for that case; then the domain of rationality of the covariants F, M and N must be specified and the theorem restated.

§9. *Affiliants and Semi-combinants in a System of more than two Quantics, and in Systems of Mixed Forms.*

In a system of more than two independent quantics of different orders in a given number of variables, those invariant concomitants are semi-combinants of the system which are semi-combinants of every pair of quantics taken separately. These must all be obtainable as concomitants of a derived system of groundquantics, where each of the given quantics, except the lowest, has been replaced by an affiliant of all quantics of lower order. If, for example, three quantics: f_l, f_m, f_n are the groundforms of the system ($l > m > n$), then these are to be replaced by

$$\begin{aligned} f_n &= f_n, \\ F_m &= f_m + f_n . L_{m,n}(f_m), \\ F_l &= f_l + F_m L_{l,m}(f_l) + f_n L_{l,n}(f_l), \end{aligned}$$

F_m being an affiliant of f_n; F_l, a simultaneous affiliant of f_n and F_m. Then it is evident, as in the case of semi-combinants of two quantics, that

Every simultaneous covariant of the three quantics f_n, F_m, F_l *is a semi-combinant of the quantics* f_n, f_m, f_l; *and conversely, that every semi-combinant of the latter set is a simultaneous covariant of the former set.*

The proof that semi-combinant groundforms are affiliants of the several quantics of lower order, and the discussion of the complete system of differential equations satisfied by semi-combinants, appear more interesting even than the corresponding parts of the foregoing, where the number of quantics has been limited to two.

So far, forms in a single set of variables have been treated. In addition to these, the theory of algebraic forms must deal with those containing two or more independent, or cogredient, or contragredient sets of variables; first of all, if we follow Clebsch, those containing two sets of the same number of variables, mutually contragredient. Such mixed forms he calls *connexes*, and speaks of them by the use of two indices, as (m_1, m_2), denoting respectively the order in the one set and the order in the other set of variables. Without going into details, I may offer here the outline statement that,

If two connexes have the indices (m_1, m_2) *of the one not less than, and at least one of them greater than the corresponding indices* (n_1, n_2), *respectively, of the other; then there exists in the aggregate of their simultaneous covariants a subgroup, their semi-combinants; and their semi-combinant groundforms and affiliants can be found by a method exactly analogous to that here given for quantics in a single set of variables.*

NORTHWESTERN UNIVERSITY, EVANSTON, ILL., *November* 30, 1894.

Simplification of Gauss's third Proof that every Algebraic Equation has a Root.

By Maxime Bôcher.

While Gauss's first and second proofs of the fundamental theorem of algebra have found many commentators, some of whom have even succeeded in essentially simplifying them, Gauss's third proof (Ges. Werke, p. 59 and p. 107), which was evidently considered by the author himself as not the least worthy of notice, is seldom referred to and, as far as I am aware, no attempt has been made to simplify it. This last fact is doubtless due to the very simplicity of the original proof which precludes any very great simplification, while its apparent failure to excite the interest of mathematicians may probably be in part explained by the fact that it appears at first sight to consist of nothing more than a skillful manipulation of formulæ. I have shown on another occasion* that Gauss's proof amounts practically to the application of a familiar theorem in the theory of the potential to the real part of the function $zf'(z)/f(z)$, where $f(z)=0$ is the equation for which we wish to prove the existence of a root. The simplified proof which follows is really equivalent to the application of the same method to the function $1/f(z)$. I have, however, followed as closely as possible the form of Gauss's proof.

We will write the equation for which we wish to prove the existence of a root in the form

$$a_0z^n + a_1z^{n-1} + \dots + a_{n-1}z + 1 = 0,$$

and we will suppose that the coefficients $a_0, a_1, \dots a_{n-1}$ are real.† Let us write $z = r(\cos\phi + i\sin\phi)$. Then the first member of the equation may be written $t + ui$ where

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* In a note recently sent to the Bulletin of the American Mathematical Society.

† Both the proof here given and Gauss's proof could easily be extended so as to cover the case of complex coefficients.

$$t = a_0 r^n \cos n\phi + a_1 r^{n-1} \cos(n-1)\phi + \ldots . + a_{n-1} r \cos\phi + 1,$$
$$u = a_0 r^n \sin n\phi + a_1 r^{n-1} \sin(n-1)\phi + \ldots . + a_{n-1} r \sin\phi .$$

We have to prove that for some pair of values of r and ϕ, t and u both vanish. Let us consider the double integral

$$\Omega = \int_{r=0}^{r=R} \int_{\phi=0}^{\phi=2\pi} \frac{(u^2 - t^2)t' - 2tuu'}{r(t^2 + u^2)^2} \, dr \, d\phi = \int_{r=0}^{r=R} \int_{\phi=0}^{\phi=2\pi} y . dr . d\phi ,$$

in which

$$t' = a_0 n r^n \cos n\phi + a_1(n-1) r^{n-1} \cos(n-1)\phi + \ldots . + a_{n-1} r \cos\phi ,$$
$$u' = a_0 n r^n \sin n\phi + a_1(n-1) r^{n-1} \sin(n-1)\phi + \ldots . + a_{n-1} r \sin\phi .$$

The function y which stands above under the signs of integration is evidently single-valued and continuous for all values of r and ϕ except such as make $t^2 + u^2 = 0$, i. e. such as make t and u vanish together, since the factor r which appears in the denominator occurs also in each term of the numerator. If, then, there were no values of r and ϕ for which t and u vanish together, we could, in computing the value of Ω, perform the two integrations in either order.

We have the following indefinite integral formulæ, as is seen by direct differentiation:

$$\int y dr = \frac{t}{t^2 + u^2}, \quad \int y d\phi = \frac{-u}{r(t^2 + u^2)} .$$

If we take this last integral between the limits $\phi = 0$ and $\phi = 2\pi$, we evidently get the value zero, since the indefinite integral vanishes at both limits. Thus by integrating first with regard to ϕ and then with regard to r, we get

$$\Omega = 0 .$$

Let us now integrate first with regard to r and then with regard to ϕ. We get, by using the first indefinite integral given above,

$$\Omega = \int_{\phi=0}^{\phi=2\pi} \left[\frac{T}{T^2 + U^2} - 1 \right] d\phi ,$$

where T and U are the values of t and u when $r = R$. We will now take R, which has so far been entirely undetermined, greater than the largest of the following positive quantities:

$$\frac{\sqrt{8}\,n\,|a_1|}{|a_0|},\quad \sqrt{\frac{\sqrt{8}\,n\,|a_2|}{|a_0|}},\quad \sqrt[3]{\frac{\sqrt{8}\,n\,|a_3|}{|a_0|}},\ldots\ldots \sqrt[n]{\frac{\sqrt{8}\,n}{|a_0|}}.^{*}$$

It is easily seen that for every value of ϕ at least one of the following inequalities will hold:

$$|T|>n,\quad |U|>n,$$

viz. the first when $|\cos n\phi|\geqq\sqrt{\tfrac{1}{2}}$, the second when $|\sin n\phi|\geqq\sqrt{\tfrac{1}{2}}$. To prove this let us indicate by ε_p, ε_p', η_p quantities numerically less than 1 except that $|\eta_1|\leqq 1$. Then we can write, when $|\cos n\phi|\geqq\sqrt{\tfrac{1}{2}}$,

$$R=\frac{\sqrt{8}\,n\,|a_1|}{|a_0|\,\varepsilon_1},\quad R^2=\frac{\sqrt{8}\,n\,|a_2|}{|a_0|\,\varepsilon_2},\ldots\ldots R^n=\frac{\sqrt{8}\,n}{|a_0|\,\varepsilon_n},$$

$$T=\frac{\sqrt{8}\,n}{|a_0|\,\varepsilon_n}\left[a_0\cos n\phi+\frac{|a_0|\,\varepsilon_1'}{\sqrt{8}\,n}+\frac{|a_0|\,\varepsilon_2'}{\sqrt{8}\,n}+\ldots\ldots+\frac{|a_0|\,\varepsilon_n}{\sqrt{8}\,n}\right]$$

$$=\frac{\sqrt{8}\,n}{|a_0|\,\varepsilon_n}\left[\frac{|a_0|}{\sqrt{2}\,\eta_1}+\frac{|a_0|\,\eta_2}{2\sqrt{2}}\right]=\frac{2n}{\varepsilon_n}\left[\frac{1}{\eta_1}+\frac{\eta_2}{2}\right]=\frac{n}{\varepsilon_n\eta_2}.$$

In the same way it can be proved that when $|\sin n\phi|\geqq\sqrt{\tfrac{1}{2}}$, $|U|>n$.

This being the case,

$$\frac{T}{T^2+U^2}-1=-\frac{T^2+U^2-T}{T^2+U^2}$$

will certainly be negative for all values of ϕ, and its integral from $\phi=0$ to $\phi=2\pi$ will be negative. Our hypothesis that there is no pair of values of r and ϕ for which t and u both vanish thus leads us to the contradiction that Ω has the value zero when computed in one way and a negative value when computed in another way. Our original equation must therefore have a root.

HARVARD UNIVERSITY, *March*, 1895.

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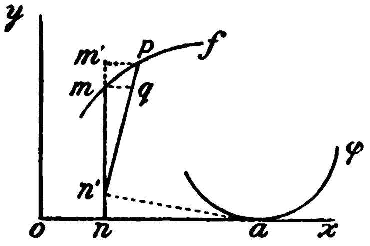
* $|a|$ means the absolute (numerical) value of a. We might easily have found smaller upper limits for R had there been an object in doing so.

Note sur les lignes cycloïdales.

Par René de Saussure.

Les formules générales relatives aux roulettes ont été données sous diverses formes, correspondant à divers systèmes de coordonnées. Il y a certains cas où l'on arrive à des équations très-simples en se servant simultanément des coordonnées cartésiennes et des coordonnées intrinsèques.

Une courbe dont l'équation intrinsèque est $\rho = \phi(\sigma)$ roule sur une droite; un point m du plan entraîné dans le mouvement décrit la trajectoire $y = f(x)$, l'axe des x étant la droite sur laquelle roule la courbe. Le problème consiste, étant donnée l'une des courbes ϕ ou f, à trouver l'autre.



Lorsque la courbe ϕ tourne d'un angle $d\alpha = \frac{d\sigma}{\rho}$, le point m vient en p, le point n en n', et l'ordonnée mn en $n'p$. L'angle $mn'p = d\alpha = \frac{\overline{mq}}{\overline{mn'}} = \frac{dx}{y}$.

Si l'on décrit l'arc de cercle $\overline{pm'}$ en prenant n' comme centre, on aura $\overline{mn} = \overline{n'p} = \overline{n'm'}$ et par suite $\overline{nn'} = \overline{mm'} = dy$. D'autre part l'angle $\overline{nan'} = d\alpha = \frac{\overline{nn'}}{\overline{na}} = \frac{dy}{\sigma - x}$. Egalant les trois valeurs trouvées pour $d\alpha$, on a finalement:

$$\frac{d\sigma}{\rho} = \frac{dy}{\sigma - x} = \frac{dx}{y},$$

équations qui résolvent le problème. Ces équations permettent de plus, étant donnée la courbe f de trouver la courbe ϕ sans effectuer de quadrature, car en les résolvant on obtient:

$$\begin{cases} \sigma = x + y\dfrac{dy}{dx}, \\ \rho = y\dfrac{d\sigma}{dx} \qquad , \end{cases}$$

et il n'y a qu'à éliminer x entre ces deux équations. Supposons par exemple que la courbe f soit une conique dont un des axes coïncide avec l'axe des x. Son équation peut se mettre sous la forme: $y^2 = ax^2 + b$, d'où l'on tire: $y\dfrac{dy}{dx} = ax$, et par suite:

$$\begin{cases} \sigma = (a+1)\,x, \\ \rho = (a+1)\,y. \end{cases}$$

Eliminant x et y, on trouve pour équation de la courbe roulante: $\rho^2 = a\sigma^2 + b\,(a+1)^2$. Lorsque la conique est une ellipse, $(a < 0)$, cette équation représente une épicycloïde; d'ailleurs dans ce cas b doit être positif. Lorsque la conique est une hyperbole, $(a > 0)$, l'équation précédente représente une famille intéressante de courbes, dont la forme varie suivant que b est négatif ou positif (c'est-à-dire suivant l'axe de l'hyperbole qui est pris comme base du roulement). Ces courbes ne semblent pas avoir été jusqu'ici considérées comme appartenant à la famille des "lignes cycloïdales"; or chaque fois que la recherche d'un lieu conduit à une ligne cycloïdale, les courbes précédentes satisfont aussi au problème, ce qui montre qu'elles ont les mêmes propriétés. C'est ainsi que Mr. Puiseux les a rencontrées en cherchant les courbes qui sont semblables à leur développée n^{me}. Mr. Laisant et Mr. Cesaro en parlent aussi conjointement avec les lignes cycloïdales, à propos d'autres propriétés, mais je crois sans les considérer comme des lignes cycloïdales. Or, elles peuvent être engendrées par le roulement d'un cercle sur un autre cercle, elles rentrent par conséquent dans la définition ordinaire des lignes cycloïdales. Cette génération, à peu près évidente, mérite pourtant d'être prise en considération et il serait désirable de donner à ces courbes des noms appropriés à leur nature. On pourrait par exemple appeler "paracycloïde" toute ligne cycloïdale pour laquelle $a > 0$ et $b < 0$ et "hypercycloïde" celle pour laquelle $a > 0$ et $b > 0$. Ainsi l'hyperbole est engendrée par le roulement d'une paracycloïde sur son axe principal ou par celui d'une hypercycloïde sur son axe conjugué.

Comme l'équation intrinsèque des lignes cycloïdales engendrées par un point d'un cercle de rayon r roulant sur un cercle fixe de rayon R est:

$$\rho^2 = -\frac{R^2}{(2r-R)^2}\sigma^2 + 16r^2\frac{(r-R)^2}{(2r-R)^2},$$

on voit que lorsque R et r sont réels la courbe est une épicycloïde. Si l'on pose $r = \frac{R}{2} + qi$ (R étant toujours réel), l'équation précédente ne contient pas d'imaginaires et de plus $a > 0$, $b < 0$; la paracycloïde est donc engendrée par le roulement d'un cercle imaginaire de rayon $\frac{R}{2} + qi$ (q étant une constante arbitraire) sur un cercle réel de rayon R.* Si l'on pose $r' = \frac{R}{2} - qi$ on retrouve la même paracycloïde; celle-ci est donc susceptible d'une double génération comme les épicycloïdes (car $r + r' = R$). La forme de la courbe est analogue à celle d'une développante de cercle; elle a un point de rebroussement qui correspond au sommet de l'hyperbole. Lorsqu'elle roule sur une droite le point qui décrit l'hyperbole est toujours le centre du cercle R. La paracycloïde admet pour asymptotes les deux spirales logarithmiques $\rho = \pm\sqrt{a}\,\sigma$, qui correspondent aux asymptotes de l'hyperbole (ces asymptotes sont imaginaires pour les épicycloïdes); enfin la normale de la paracycloïde est toujours extérieure au cercle R, le centre instantané des deux cercles devant être imaginaire. Lorsque $b = 0$, l'hyperbole se réduit aux droites $y = \pm\sqrt{a}\,x$ et la paracycloïde dégénère en spirales logarithmiques $\rho = \pm\sqrt{a}\,\sigma$.

Si maintenant l'on pose $r = p + \frac{R}{2}i$ (le rayon du cercle fixe étant Ri et p étant une constante arbitraire) les coefficients de l'équation intrinsèque sont encore réels et de plus $a > 0$, $b > 0$; l'hypercycloïde est donc engendrée par le roulement d'un cercle imaginaire sur un cercle imaginaire fixe dont le centre est réel. C'est toujours ce centre qui décrit l'hyperbole. En posant $r' = -p + \frac{R}{2}i$ on trouve la même hypercycloïde et par conséquent le même théorème sur la double génération (car $r + r' = Ri$). La courbe a toujours la forme de spirale, mais le point de rebroussement a disparu. Elle admet les mêmes asymptotes que la paracycloïde. Enfin la développée de l'hypercycloïde est une paracycloïde et réciproquement.

---

* Il est facile de voir comment une courbe imaginaire peut en roulant sur une courbe réelle (ou imaginaire) engendrer une courbe réelle, car pour définir un déplacement du plan, il faut définir les déplacements de deux de ses points. Donc un point unique peut décrire une trajectoire réelle sans que pour cela le déplacement du plan soit nécessairement réel; mais si le déplacement est imaginaire, il n'y aura jamais plus d'un point du plan décrivant une courbe réelle.

Si la conique donnée est une parabole $y^2 = 2px$, on trouvera par la même méthode que la courbe roulante est une développante de cercle $\rho^2 = 2p\sigma$, c'est-à-dire encore une ligne cycloïdale. Ainsi toute conique peut être considérée comme engendrée par le roulement sur l'un quelconque de ses axes d'une ligne cycloïdale; le point décrivant la conique est toujours le centre du cercle R.

Il peut être utile de remarquer que les quantités ρ et σ sont les coordonnées cartésiennes du centre de courbure de la courbe roulante, correspondant au point de contact, puisque σ est égale à l'abcisse de ce point de contact. Donc les équations données plus haut qui font dépendre x, y de ρ et σ, permettent de trouver directement le lieu de ces centres de courbure lorsque la courbe f est donnée ou réciproquement. Pour les coniques on a trouvé que σ et ρ sont proportionnels à x et y, le lieu des centres de courbure est donc une conique semblable.

Note on Lines of Curvature.

By Thomas Hardy Taliaferro.

In a note in the Comptes Rendus for March 25th, 1895, Professor Craig has given a condition for the determination of surfaces having lines of curvature corresponding to a system of conjugate lines on a given surface.

Suppose the surface to be represented by the equations

$$\begin{cases} x = f_1(\rho, \rho_1), \\ y = f_2(\rho, \rho_1), \\ z = f_3(\rho, \rho_1), \end{cases}$$

where ρ, ρ_1 are the parameters of a system of conjugate lines; then in the note referred to it is shown how surfaces can be found whose coordinates are given by

$$X = \psi_1(x), \quad Y = \psi_2(y), \quad Z = \psi_3(z),$$

on which the original conjugate lines are lines of curvature.

The condition to be satisfied is of course

$$\frac{\partial X}{\partial \rho}\frac{\partial X}{\partial \rho_1} + \frac{\partial Y}{\partial \rho}\frac{\partial Y}{\partial \rho_1} + \frac{\partial Z}{\partial \rho}\frac{\partial Z}{\partial \rho_1} = 0. \tag{1}$$

The first difficulty in the problem is in finding an initial surface whose coordinates are given explicitly as functions of the parameters of a system of conjugate lines. Certain methods are known for this, especially the elegant method of Koenigs (Darboux, Vol. I, page 112), but all are very difficult of application in any particular case.

I have ventured in the following brief note to give a simple application of the problem to the case of tetrahedral surfaces where $m = n$, and also to give two examples. The tetrahedral surfaces are given by the equations (Darboux, Vol. I, page 142):

$$\left.\begin{aligned} x &= \lambda A(\rho - a)^m(\rho_1 - a)^n, \\ y &= \mu B(\rho - b)^m(\rho_1 - b)^n, \\ z &= \nu C(\rho - c)^m(\rho_1 - c)^n, \end{aligned}\right\} \tag{2}$$

where ρ, ρ_1 are the parameters of a system of conjugate lines; m, n, A, B, C any real constants; and λ, μ, ν either 1 or i.

The left-hand members of (2) all satisfy the equation

$$(\rho_1-\rho)\frac{\partial^2\theta}{\partial\rho\,\partial\rho_1}+n\frac{\partial\theta}{\partial\rho}-m\frac{\partial\theta}{\partial\rho_1}=0, \tag{3}$$

since ρ, ρ_1 are the parameters of a system of conjugate lines, but as $x^2+y^2+z^2$ does not satisfy equation (3), ρ, ρ_1 are not the parameters of the lines of curvature. It is readily seen that the condition to be satisfied in order that ρ, ρ_1 should be the parameters of the lines of curvature is

$$\lambda^2A^2(\rho-a)^{2m-1}(\rho_1-a)^{2n-1}+\mu^2B^2(\rho-b)^{2m-1}(\rho_1-b)^{2n-1} \\ +\nu^2C^2(\rho-c)^{2m-1}(\rho_1-c)^{2n-1}=0. \tag{4}$$

I.—*Tetrahedral Surfaces, when* $m=n$.

When $m=n$, the expressions for the cartesian coordinates x, y, z of the tetrahedral surfaces in terms of the parameters ρ, ρ_1 of a system of conjugate lines become

$$\left.\begin{aligned} x&=\lambda A(\rho-a)^m(\rho_1-a)^m,\\ y&=\mu B(\rho-b)^m(\rho_1-b)^m,\\ z&=\nu C(\rho-c)^m(\rho_1-c)^m.\end{aligned}\right\} \tag{5}$$

The equation of condition (4) becomes

$$\lambda^2A^2(\rho-a)^{2m-1}(\rho_1-a)^{2m-1}+\mu^2B^2(\rho-b)^{2m-1}(\rho_1-b)^{2m-1} \\ +\nu^2C^2(\rho-c)^{2m-1}(\rho_1-c)^{2m-1}=0. \tag{6}$$

The equation of the tetrahedral surface on eliminating ρ, ρ_1 is readily seen to be of the form

$$\alpha\left(\frac{x}{\lambda A}\right)^{1/m}-\beta\left(\frac{y}{\mu B}\right)^{1/m}+\gamma\left(\frac{z}{\nu C}\right)^{1/m}=1, \tag{7}$$

where

$$\left\{\begin{aligned} \alpha&=\frac{1}{(a-b)(a-c)},\\ \beta&=\frac{1}{(a-b)(b-c)},\\ \gamma&=\frac{1}{(b-c)(a-c)},\end{aligned}\right.$$

$$\alpha-\beta+\gamma=0.$$

On adopting the convention

$$a > b > c,$$

it is seen that α, β, γ are real, positive quantities fulfilling the condition

$$\alpha < \beta < \gamma.$$

I say that writing

$$X = k_1 x^{1/2m}, \quad Y = k_2 y^{1/2m}, \quad Z = \Phi(z),$$

where k_1, k_2 are arbitrary constants, $\Phi(z)$ can be so determined by means of Craig's formula that on the derived surface ρ, ρ_1 will be the parameters of the lines of curvature, and furthermore that the derived surface will be a quadric surface depending on A, B, C, λ, μ, ν, k_1, k_2 for its form.

Since equation (1) consists of a single equation between three quantities, two of them may be assumed and the third determined.

Let

$$\psi_1(x) = k_1 x^{1/2m}, \quad \psi_2(y) = k_2 y^{1/2m}, \quad \psi_3(z) = \Phi(z), \tag{8}$$

where $\Phi(z)$ is to be determined.

Substituting these values (8) in equation (1), the following equation is derived:

$$\frac{1}{4}(\lambda A)^{1/m} k_1^2 + \frac{1}{4}(\mu B)^{1/m} k_2^2 + m^2 \nu^2 C^2 (\rho - c)^{2m-1} (\rho_1 - c)^{2m-1} \left(\frac{d\Phi}{dz}\right)^2 = 0,$$

$$d\Phi = \frac{\pm i}{2m\nu C} \{(\lambda A)^{1/m} k_1^2 + (\mu B)^{1/m} k_2^2\}^{\frac{1}{2}} \frac{dz}{\{(\rho - c)^{2m-1} (\rho_1 - c)^{2m-1}\}^{\frac{1}{2}}},$$

$$dz = \frac{\partial z}{\partial \rho} d\rho + \frac{\partial z}{\partial \rho_1} d\rho_1,$$

$$\therefore \quad d\Phi = \pm i \{(\lambda A)^{1/m} k_1^2 + (\mu B)^{1/m} k_2^2\}^{\frac{1}{2}} d\{(\rho - c)^{\frac{1}{2}} (\rho_1 - c)^{\frac{1}{2}}\},$$

$$\therefore \Phi(z) = \frac{\pm i z^{1/2m}}{(\mu C)^{1/2m}} \{(\lambda A)^{1/m} k_1^2 + (\mu B)^{1/m} k_2^2\}^{\frac{1}{2}}. \tag{9}$$

The derived surface has for its cartesian coordinates X, Y, Z the following expressions:

$$\left.\begin{aligned} X &= k_1 (\lambda A)^{1/2m} (\rho - a)^{\frac{1}{2}} (\rho_1 - a)^{\frac{1}{2}}, \\ Y &= k_2 (\mu B)^{1/2m} (\rho - b)^{\frac{1}{2}} (\rho_1 - b)^{\frac{1}{2}}, \\ Z &= \pm i \{k_1^2 (\lambda A)^{1/m} + k_2^2 (\mu B)^{1/m}\}^{\frac{1}{2}} (\rho - c)^{\frac{1}{2}} (\rho_1 - c)^{\frac{1}{2}}. \end{aligned}\right\} \tag{10}$$

Equations (1) and (3) are satisfied, and the equation of the derived surface on which ρ, ρ_1 are the parameters of the lines of curvature is

$$a\frac{X^2}{k_1^2(\lambda A)^{1/m}}-\beta\frac{Y^2}{k_2^2(\mu B)^{1/m}}-\gamma\frac{Z^2}{k_1^2(\lambda A)^{1/m}+k_2^2(\mu B)^{1/m}}=1. \tag{11}$$

Equation (11) is the equation of a quadric surface depending on A, B, C, λ, μ, ν, k_1, k_2 for its form.

II.—*Examples.*

1. $m=n=\frac{1}{2}$, $\lambda=\nu=1$, $\mu=i$.

Equations (5) become

$$\begin{cases} x= A(\rho-a)^{\frac{1}{2}}(\rho_1-a)^{\frac{1}{2}}, \\ y=iB(\rho-b)^{\frac{1}{2}}(\rho_1-b)^{\frac{1}{2}}, \\ z= C(\rho-c)^{\frac{1}{2}}(\rho_1-c)^{\frac{1}{2}}. \end{cases}$$

Equation (7) becomes

$$a\frac{X^2}{A^2}+\beta\frac{y^2}{B^2}+\gamma\frac{z^2}{C^2}=1,$$

which is the equation of an ellipsoid.

The equation of condition (6) that ρ, ρ_1 should be lines of curvature on the original surface reduces in the case of the ellipsoid to

$$A^2-B^2+C^2=0.$$

Equation (8) becomes

$$\psi_1(x)=k_1x,\quad \psi_2(y)=k_2y,\quad \psi_3(z)=\Phi(z),$$
$$\Phi(z)=\pm\frac{(k_2^2B^2-k_1^2A^2)^{\frac{1}{2}}z}{C}.$$

The derived surface has for its cartesian coordinates X, Y, Z the following expressions:

$$\begin{cases} X= k_1A(\rho-a)^{\frac{1}{2}}(\rho_1-a)^{\frac{1}{2}}, \\ Y=ik_2B(\rho-b)^{\frac{1}{2}}(\rho_1-b)^{\frac{1}{2}}, \\ Z=\pm(k_2^2B^2-k_1^2A^2)^{\frac{1}{2}}(\rho-c)^{\frac{1}{2}}(\rho_1-c)^{\frac{1}{2}}. \end{cases}$$

The equation of the derived surface becomes

$$a\frac{X^2}{k_1^2A^2}+\beta\frac{Y^2}{k_2^2B^2}+\gamma\frac{Z^2}{k_2^2B^2-k_1^2A^2}=1,$$

which is a quadric surface.

Making certain suppositions on k_1, k_2 the following surfaces are derived:

$$k_1 = 1,\ k_2 = 1, \qquad \alpha\frac{X^2}{A^2} + \beta\frac{Y^2}{B^2} + \gamma\frac{Z^2}{B^2 - A^2} = 1,\ B > A, \text{ an ellipsoid;}$$

$$k_1 = i,\ k_2 = 1, \quad -\alpha\frac{X^2}{A^2} + \beta\frac{Y^2}{B^2} + \gamma\frac{Z^2}{B^2 + A^2} = 1, \text{ an hyperboloid of one sheet;}$$

$$k_1 = 1,\ k_2 = i, \qquad \alpha\frac{X^2}{A^2} - \beta\frac{Y^2}{B^2} - \gamma\frac{Z^2}{B^2 + A^2} = 1, \text{ an hyperboloid of two sheets.}$$

Writing the equation of condition (1) for the case of the ellipsoid, it becomes

$$\left(\frac{d\psi_1}{dx}\right)^2 A^2 - \left(\frac{d\psi_2}{dy}\right)^2 B^2 + \left(\frac{d\psi_3}{dz}\right)^2 C^2 = 0. \tag{12}$$

On writing

$$\frac{d\psi_1}{dx} = BC, \quad \frac{d\psi_2}{dy} = \sqrt{2}\,CA, \quad \frac{d\psi_3}{dz} = AB,$$

equation (12) is satisfied and the values of ψ_1, ψ_2, ψ_3 are

$$\psi_1(x) = BCx, \quad \psi_2(y) = \sqrt{2}\,CAy, \quad \psi_3(z) = ABz. \tag{13}$$

The expressions for the cartesian coordinates X, Y, Z of the derived surface on which ρ, ρ_1 are the parameters of the lines of curvature are

$$\left.\begin{aligned} X &= D(\rho - a)^{\frac{1}{2}}(\rho_1 - a)^{\frac{1}{2}}, \\ Y &= \sqrt{2}\,iD(\rho - b)^{\frac{1}{2}}(\rho_1 - b)^{\frac{1}{2}}, \\ Z &= D(\rho - c)^{\frac{1}{2}}(\rho_1 - b)^{\frac{1}{2}}, \end{aligned}\right\} \tag{14}$$

$$D = ABC.$$

The equation of the derived surface is represented by

$$\alpha\frac{X^2}{D^2} + \beta\frac{Y^2}{2D^2} + \gamma\frac{Z^2}{D^2} = 1, \tag{15}$$

which is an ellipsoid.

On writing $\lambda = 1$, $\mu = \nu = i$, the initial equation becomes an hyperboloid of one sheet, and for $\lambda = \mu = 1$, $\nu = i$, the initial equation is that of an hyperboloid of two sheets. In both cases there can be derived, as in case of ellipsoid, quadric surfaces depending on k_1, k_2, A, B, C for their form.

2. $m = n = 2,\ \lambda = \mu = \nu = 1.$

Equations (5) become

$$\begin{cases} x = A(\rho - a)^2 (\rho_1 - a)^2, \\ y = B(\rho - b)^2 (\rho_1 - b)^2, \\ z = C(\rho - c)^2 (\rho_1 - c)^2. \end{cases}$$

Equation (7) becomes

$$\alpha \left(\frac{x}{A}\right)^{\frac{1}{2}} - \beta \left(\frac{y}{B}\right)^{\frac{1}{2}} + \gamma \left(\frac{z}{C}\right)^{\frac{1}{2}} = 1,$$

which is the equation of Steiner's surface.

The equation of condition (6) that ρ, ρ_1 should be the parameters of the lines of curvature on the original surface reduces in the case of Steiner's surface to

$$A^2(\rho - a)^3 (\rho_1 - a)^3 + B^2 (\rho - b)^3 (\rho_1 - b)^3 + C^2 (\rho - c)^3 (\rho_1 - c)^3 = 0.$$

Equation (8) becomes

$$\psi_1(x) = k_1 x^{\frac{1}{4}}, \quad \psi_2(y) = k_2 y^{\frac{1}{4}}, \quad \psi_3(z) = \Phi(z),$$
$$\Phi(z) = \pm \frac{i z^{\frac{1}{4}}}{C^{\frac{1}{4}}} \sqrt{\sqrt{A} + \sqrt{B}}.$$

The derived surface has for its cartesian coordinates X, Y, Z the following expressions:

$$\begin{cases} X = k_1 A^{\frac{1}{4}} (\rho - a)^{\frac{1}{2}} (\rho_1 - a)^{\frac{1}{2}}, \\ Y = k_2 B^{\frac{1}{4}} (\rho - b)^{\frac{1}{2}} (\rho_1 - b)^{\frac{1}{2}}, \\ Z = \pm i \left\{ k_1^2 A^{\frac{1}{2}} + k_2^2 B^{\frac{1}{2}} \right\}^{\frac{1}{2}} (\rho - c)^{\frac{1}{2}} (\rho_1 - c)^{\frac{1}{2}}. \end{cases}$$

The equation of the derived surface becomes

$$\alpha \frac{X^2}{k_1^2 A^{\frac{1}{2}}} - \beta \frac{Y^2}{k_2 B^{\frac{1}{2}}} - \gamma \frac{Z^2}{k_1^2 A^{\frac{1}{2}} + k_2^2 B^{\frac{1}{2}}} = 1,$$

which is a quadric surface.

Making certain suppositions on k_1, k_2, the following surfaces are derived:

$k_1 = 1,\ k_2 = i,\quad \alpha \dfrac{X^2}{A^{\frac{1}{2}}} + \beta \dfrac{Y^2}{B^{\frac{1}{2}}} + \gamma \dfrac{Z^2}{B^{\frac{1}{2}} - A^{\frac{1}{2}}} = 1,\ B > A$ an ellipsoid;

$k_1 = i,\ k_2 = i,\ -\alpha \dfrac{X^2}{A^{\frac{1}{2}}} + \beta \dfrac{Y^2}{B^{\frac{1}{2}}} + \gamma \dfrac{Z^2}{B^{\frac{1}{2}} + A^{\frac{1}{2}}} = 1,$ an hyperboloid of one sheet;

$k_1 = 1,\ k_2 = 1,\quad \alpha \dfrac{X^2}{A^{\frac{1}{2}}} - \beta \dfrac{Y^2}{B^{\frac{1}{2}}} - \gamma \dfrac{Z^2}{B^{\frac{1}{2}} + A^{\frac{1}{2}}} = 1,$ an hyperboloid of two sheets.

Writing the equation of condition (1) for the case of Steiner's surface, it becomes

$$A^2 = \left(\frac{d\psi_1}{dx}\right)^2 (\rho - a)^3 (\rho_1 - a)^3 + B^2 \left(\frac{d\psi_2}{dy}\right)^2 (\rho - b)^3 (\rho_1 - b)^3 \\ + C^2 \left(\frac{d\psi_3}{dz}\right)^2 (\rho - c)^3 (\rho_1 - c)^3 = 0. \quad (16)$$

On writing

$$\frac{d\psi_1}{dx} = A^{\frac{1}{2}} B C x^{-\frac{3}{4}}, \quad \frac{d\psi_2}{dy} = \sqrt{2}\, i A B^{\frac{1}{2}} C y^{-\frac{3}{4}}, \quad \frac{d\psi_3}{dz} = A B C^{\frac{1}{2}} z^{-\frac{3}{4}},$$

equation (16) is satisfied and the values of ψ_1, ψ_2, ψ_3 are

$$\psi_1(x) = 4A^{\frac{1}{2}} B C x^{\frac{1}{4}}, \quad \psi_2(y) = 4\sqrt{2}\, i A B^{\frac{1}{2}} C y^{\frac{1}{4}}, \quad \psi_3(z) = 4 A B C^{\frac{1}{2}} z^{\frac{1}{4}}. \quad (17)$$

The expressions for the cartesian coordinates X, Y, Z of the derived surface on which ρ, ρ_1 are the parameters of the lines of curvature are

$$\left.\begin{aligned} X &= D(\rho - a)^{\frac{1}{2}} (\rho_1 - a)^{\frac{1}{2}}, \\ Y &= \sqrt{2}\, i D(\rho - b)^{\frac{1}{2}} (\rho_1 - b)^{\frac{1}{2}}, \\ Z &= D(\rho - c)^{\frac{1}{2}} (\rho_1 - c)^{\frac{1}{2}}, \end{aligned}\right\} \quad (18)$$

$$D = 4ABC.$$

The equation of the derived surface is represented by

$$\alpha \frac{X^2}{D^2} + \beta \frac{Y^2}{2D^2} + \gamma \frac{Z^2}{D^2} = 1, \quad (19)$$

which is an ellipsoid.

In the case of quadric surfaces, and also of Steiner's surface, there can be derived other quadric surfaces by assuming any other two of the ψ's and determining the remaining one as above.

It is also readily seen that it is necessary and sufficient for X, Y, Z to have the following values:

$$X = k_1 x^{1/2m}, \quad Y = k_2 y^{1/2m}, \quad Z = \Phi(z),$$

in order that $d\Phi(z)$ should be an exact differential in the case of tetrahedral surfaces when $m = n$.

For write $$X = k_1 x^{t/m}, \quad Y = k_2 y^{t/m}, \quad Z = \Phi(z), \quad (20)$$

where t is any constant, then the following expression for $d\Phi(z)$ is derived from equation (1):

$$d\Phi(z) = \pm\, it\,\{k_1^2(\lambda A)^{2t/m}(\rho - a)^{2t-1}(\rho_1 - a)^{2t-1} \\ + k_2^2(\mu B)^{2t/m}(\rho - b)^{2t-1}(\rho_1 - b)^{2t-1}\}^{\frac{1}{2}}\left\{\left(\frac{\rho_1 - c}{\rho - c}\right)^{\frac{1}{2}} d\rho + \left(\frac{\rho - c}{\rho_1 - c}\right)^{\frac{1}{2}} d\rho_1\right\}. \quad (21)$$

Write

$$it\,\{k_1^2(\lambda A)^{2t/m}(\rho - a)^{2t-1}(\rho_1 - a)^{2t-1} + k_2^2(\mu B)^{2t/m}(\rho - b)^{2t-1}(\rho_1 - b)^{2t-1}\}^{\frac{1}{2}} = U, \quad (22)$$

the condition for the integrability of equation (21) is

$$\frac{\partial}{\partial \rho_1} U\left(\frac{\rho_1 - c}{\rho - c}\right)^{\frac{1}{2}} - \frac{\partial}{\partial \rho} U\left(\frac{\rho - c}{\rho_1 - c}\right)^{\frac{1}{2}} = 0, \quad (23)$$

or

$$\left(\frac{\rho_1 - c}{\rho - c}\right)^{\frac{1}{2}} \frac{\partial U}{\partial \rho_1} - \left(\frac{\rho - c}{\rho_1 - c}\right)^{\frac{1}{2}} \frac{\partial U}{\partial \rho} = 0.$$

Since in general

$$\frac{\rho_1 - c}{\rho - c} \neq \frac{\rho - c}{\rho_1 - c},$$

it is necessary, in order to satisfy equation (23), that

$$\frac{\partial U}{\partial \rho} = 0, \quad \frac{\partial U}{\partial \rho_1} = 0,$$

or

$$\left.\begin{aligned} \frac{\partial}{\partial \rho}\, it\,\{k_1^2(\lambda A)^{2t/m}(\rho - a)^{2t-1}(\rho_1 - a)^{2t-1} + k_2^2(\mu B)^{2t/m}(\rho - b)^{2t-1}(\rho_1 - b)^{2t-1}\}^{\frac{1}{2}} = 0, \\ \frac{\partial}{\partial \rho_1}\, it\,\{k_1^2(\lambda A)^{2t/m}(\rho - a)^{2t-1}(\rho_1 - a)^{2t-1} + k_2^2(\mu B)^{2t/m}(\rho - b)^{2t-1}(\rho_1 - b)^{2t-1}\}^{\frac{1}{2}} = 0, \end{aligned}\right\} (24)$$

which requires

$$it\{k_1^2(\lambda A)^{2t/m}(\rho - a)^{2t-1}(\rho_1 - a)^{2t-1} + k_2^2(\mu B)^{2t/m}(\rho - b)^{2t-1}(\rho_1 - b)^{2t-1}\}^{\frac{1}{2}} = \text{const.}, \quad (25)$$

for which it is necessary and sufficient that $t = \frac{1}{2}$. Hence

$$X = k_1 x^{1/2m}, \quad Y = k_2 y^{1/2m}, \quad Z = \Phi(z).$$

Other special cases may arise where $t \neq \frac{1}{2}$, but they will obviously require some relation to exist between the constants in equation (25).

JOHNS HOPKINS UNIVERSITY, *May 1st*, 1895.

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On the Deformation of Thin Elastic Wires.

By A. B. Basset, M. A., F. R. S.

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1. An account of the various attempts which have been made to construct a theory of the deformation of a thin elastic wire, together with the solution of various problems of interest, will be found in the second volume of Mr. Love's recent Treatise on Elasticity. The above work also contains a variety of *geometrical* investigations connected with this subject, and the methods employed are of considerable novelty, power and elegance. But Mr. Love's treatment of the *physical* portion of the subject is not at all so satisfactory; and this is in great measure due to the fact, which I have commented upon in my recent paper on the Deformation of Thin Elastic Plates and Shells,* that he appears to entertain some objection against the method of expansion, and has also been unable to emancipate himself from the imperfect methods of the German and French schools.

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* Amer. Journ. of Math., Vol. XVI, p. 255.

I am of opinion that the most satisfactory way of constructing a complete theory of the small deformations of thin wires is to employ the method explained in my paper on the Theory of Elastic Wires;[*] but unfortunately that investigation contains a slight slip in the work, which arose from my having copied an equation wrongly and used the wrong equation in a subsequent portion of the paper. In consequence of this, the values of the two flexural couples are not proportional to the changes of curvature, as ought to be the case. This circumstance may possibly have led Mr. Love to entertain doubts as to the soundness of the principles upon which the theory was based; whereas the real fact is that the theory is a perfectly sound and unimpeachable one, and when the error is corrected it leads to results which have been established by methods of a more or less imperfect character, which agree with those obtained by Mr. Love, and are now generally admitted to be correct.

Under these circumstances I think that a further exposition of the theory of wires is needed, and this is what I propose to give in the present paper. I shall commence with the theory of the small deformations of a naturally curved wire; I shall then discuss the theory of finite deformations, in which finite changes of curvature and twist occur; and I shall lastly work out the solutions of various problems of interest.

In most problems of practical interest, the wire is made of flexible and well-tempered metal such as steel; also its cross-section is uniform and circular, and the radius of the latter is small in comparison with the radius of principal curvature of the wire at any point of its length. Wires of this description are called *thin wires*, and to such wires the following investigation will be exclusively confined. It is also obvious that the central axis of a metal wire may usually be regarded as inextensible, since any extension of the axis which might possibly be produced by any given forces is extremely small in comparison with the flexion and torsion actually produced. We shall therefore suppose that the extension of the central axis may be neglected.

The General Equations of Equilibrium.

2. When a thin wire, whose natural form is any curve, is deformed, the lines which before deformation coincided with the principal normal, the binormal and the tangent to the central axis will not usually coincide after deformation with the principal normal, the binormal and the tangent to the deformed central axis;

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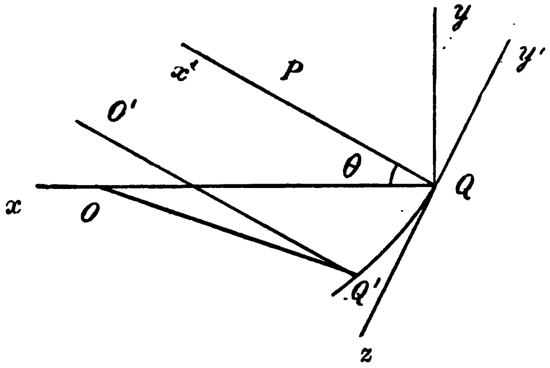
[*] Proc. Lond. Math. Soc., Vol. XXIII, p. 105.

and as our object is to obtain equations which are applicable to a wire finitely deformed, it is convenient to choose as our axis of reference the above-mentioned lines in the deformed wire. If, however, the deformation is small, it is immaterial whether the axes are supposed to refer to the deformed or the undeformed central axis, since any error which might be introduced would be of the second order of small quantities.

The resultant stresses which act across any transverse section of the deformed wire are six in number, and consist of

T = a tension along the tangent to the central axis,
N_1 = a shearing stress along the principal normal,
N_2 = a shearing stress along the binormal,
H = a torsional couple about the tangent,
G_1 = a flexural couple about the principal normal,
G_2 = a flexural couple about the binormal.

To obtain the equations of equilibrium, let Q be any point on the central axis; Qx, Qy, Qz the principal normal, binormal and tangent to the central axis at Q. Let Q' be any point on the central axis near Q; O, O' the centres of principal curvature at Q and Q'; let $\delta\phi$, $\delta\eta$ be the angles of contingence and torsion at Q, so that $QOQ' = \delta\phi$, $OQ'O' = \delta\eta$; let ρ, σ be the radii of principal curvature and torsion at Q. Also let $T + \delta T$, $N_1 + \delta N_1$, etc., be the values of the resultant stresses at Q', and $\mathfrak{X}$, $\mathfrak{Y}$, $\mathfrak{Z}$; $\mathfrak{L}$, $\mathfrak{M}$, $\mathfrak{N}$ the components of the bodily forces and couples per unit of length of the wire.



The equations of equilibrium are obtained by resolving all the forces and couples which act upon the element δs, parallel to Qz, Qx, Qy. We thus obtain

$$\left.\begin{aligned}\frac{dT}{ds}-\frac{N_1}{\rho}+\mathfrak{Z}\qquad&=0,\\ \frac{dN_1}{ds}-\frac{N_2}{\sigma}+\frac{T}{\rho}+\mathfrak{X}\qquad&=0,\\ \frac{dN_2}{ds}+\frac{N_1}{\sigma}+\mathfrak{Y}\qquad&=0,\\ \frac{dH}{ds}-\frac{G_1}{\rho}+\mathfrak{N}\qquad&=0,\\ \frac{dG_1}{ds}-\frac{G_2}{\sigma}+\frac{H}{\rho}-N_2+\mathfrak{L}&=0,\\ \frac{dG_2}{ds}+\frac{G_1}{\sigma}+N_1+\mathfrak{M}&=0.\end{aligned}\right\}\qquad(1)$$

In statical problems the three couples $\mathfrak{L}$, $\mathfrak{M}$, $\mathfrak{N}$ are usually zero; but in dynamical problems they must be replaced by the time variations (taken with the negative sign) of the components of the angular momentum of the element.

Equations (1) in their present form do not enable us to solve any statical or dynamical problems; in order to do this we require the values of the three couples. We shall hereafter show that the flexural couples are proportional to the changes of curvature, whilst the torsional couple is proportional to the change of twist; and these theorems combined with (1) are sufficient to enable us to solve a variety of problems relating to finite deformations. When the deformation is small, the values of the three couples (and consequently the changes of curvature and twist) can be expressed in terms of the displacements of the point Q, together with a certain angle β which is connected with the twist; and these results combined with the condition of inextensibility will furnish a sufficient number of equations for the solution of every problem.

Theory of Small Deformations.

3. In the theory of thin plates and shells, the three stresses R, S, T vanish at the free surfaces of the shell, *provided the latter are not subjected to any surface pressures or tangential stresses;* and I have shown in my previous papers that, subject to this limitation, the terms of lowest order which these stresses contain are quadratic functions of h and h', where $2h$ is the thickness of the shell and h' is the distance of a point in its substance from the middle surface. The coefficients of h and h' in these quadratic functions are unknown quantities which

cannot be determined by any direct method; but I have shown that if the investigation is confined to an approximate solution which does not involve higher powers of the thickness than the cube, it is not necessary to ascertain the values of these unknown quantities; in other words, *the three stresses which vanish at the surface may be treated as zero.* Under these circumstances it appeared to me that the most natural and appropriate course was to employ a similar hypothesis as the basis of the theory of wires.

In the figure let P be *any* point in the cross-section through Q, and let Qx', Qy', Qz be a subsidiary set of rectangular axes of which the axis of x' passes through P; also let P, Q, R, S, T, U be the six components of stress at P referred to this subsidiary set of axes. Since the cross-section of the wire is circular, the three stresses P, T, U must vanish at the surface provided the latter is free from stress; and from analogy to the theory of thin plates it is natural to suppose that within the substance of the wire these stresses are *small quantities which may be treated as zero* provided the solution is confined to a certain degree of approximation. And I found that if the terms of lowest order in P, T, U were quadratic functions of c and r, where c is the radius of the cross-section and $QP = r$, the values of the three couples could be calculated as far as the *fourth* power of the radius by treating P, T, U as zero, since the retention of these quantities would lead to terms involving higher powers of c than the fourth. I accordingly based the theory on the following fundamental hypothesis:

The three stresses P, T, U are small quantities which may be treated as zero, provided the surface of the wire is not subjected to any surface forces such as pressures or tangential stresses; and provided also that the approximate expressions for the energy and couples do not include any higher powers of the radius of the cross-section than the fourth.

This hypothesis may possibly appear a bold one, especially as I was unable to bring forward in support of it evidence furnished by the general equations of elasticity of the same character as can be produced in the case of the corresponding hypothesis which forms the foundation of the theory of thin plates and shells; but the results to which this hypothesis lead conclusively establish its correctness.

4. The development of the theory of wires has been retarded by an erroneous assumption of Saint-Venant, that the three stresses P, Q, U are *accurately* zero.

Saint-Venant made this hypothesis the basis of his theory of the torsion of prisms, and it is remarkable that he was thereby led to results which are undoubtedly correct when the prism is *infinitely long*. I have considered this theory in my paper on wires,* and have shown on page 125 that all the results can be obtained without the aid of this highly objectionable hypothesis. The fallacy of writers who have followed Saint-Venant lies in the fact that they have imagined that a hypothesis which happens to be true in a class of problems of a very special character, can be made the basis of a general theory of wires. It can be shown that when the cross-section is circular, Q will vanish to a certain order of approximation provided P does; consequently if P may be treated as zero, Q may also be so treated. But a result which drops out incidentally in the course of the work is a totally different thing from an assumption which dogmatically asserts that the result is true; and the objection to assuming that Q may be treated as zero lies in the fact that, since it does not vanish at the boundary, no valid reason can be assigned for supposing that in the interior of the wire it is a small quantity which may be neglected.

The fact that the stress Q may not in general be treated as zero, unless the cross-section is circular, may be seen by considering the case of a wire of elliptic cross-section. If the ellipse be supposed to degenerate into two infinite parallel straight lines, the wire will become a thin plate, and the stress P in the theory of wires becomes the stress R in the theory of plates; whilst the stress Q in the theory of wires becomes the stress P (or Q) in the theory of plates. Since P (or Q) in the theory of plates may not be treated as zero, it follows that the stress Q in the theory of wires may not in general be so treated.

5. The mathematical development of the fundamental hypothesis and the procedure employed for calculating the values of the three couples are so fully explained on pages 108 to 116 of my paper on the Theory of Wires previously referred to, that it will be unnecessary to reproduce the investigation. I shall therefore proceed to show how the error I have alluded to arose, and how it is to be corrected.

The strain g is correctly given by equation (11) of that paper, but in copying out equations (13) to (18) on p. 111 a term has been omitted in (15). The correct equation is

$$g = \frac{1}{\rho - r\cos\theta}\left(\frac{dw'}{d\phi} - \frac{\rho}{\sigma}\,\frac{dw'}{d\theta} - u'\cos\theta + v'\sin\theta\right). \tag{2}$$

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* Proc. Lond. Math. Soc., Vol. XXIII.

The value of w' is correctly given by equation (31), and when the omitted term is supplied in equation (32) it will be found to lead to exactly the same values of the two flexural couples as those given by Mr. Love,* which are, as he has shown, proportional to the changes of curvature. The value of the torsional couple given by myself is quite correct, and it is proportional to the change of twist.

For brevity write

$$\lambda = \frac{du}{ds} + \frac{w}{\rho} - \frac{v}{\sigma} \quad , \quad \mu = \frac{dv}{ds} + \frac{u}{\sigma}, \tag{3}$$

$$\mathfrak{P} = \frac{d\lambda}{ds} - \frac{\mu}{\sigma} + \frac{\sigma_1 - \sigma_3}{\rho}, \quad \mathfrak{Q} = \frac{d\mu}{ds} + \frac{\lambda}{\sigma} - \frac{\beta}{\rho}, \tag{4}$$

where u, v, w are the displacements of a point on the central axis along the principal normal, binormal and tangent, and we shall obtain the following equations:

$$\left.\begin{aligned} u' &= r\sigma_1 + u\cos\theta + v\sin\theta + \frac{(m-n)\,r^3}{4m}(\mathfrak{P}\cos\theta + \mathfrak{Q}\sin\theta), \\ v' &= r\beta + v\cos\theta - u\sin\theta + \frac{(m-n)\,r^3}{4m}(\mathfrak{P}\sin\theta - \mathfrak{Q}\cos\theta), \end{aligned}\right\} \tag{5}$$

$$e = f = \sigma_1 + \frac{(m-n)\,r}{2m}(\mathfrak{P}\cos\theta + \mathfrak{Q}\sin\theta), \tag{6}$$

$$g = \sigma_3 - r(\mathfrak{P}\cos\theta + \mathfrak{Q}\sin\theta), \tag{7}$$

$$\sigma_1 = \sigma_2 = -\frac{m-n}{2m}\sigma_3. \tag{8}$$

From equations (6) and (7) it follows that $P = Q$, and since P has been assumed to be a small quantity which may be treated as zero to the order of approximation adopted, it follows that Q may also be treated as zero; accordingly Saint-Venant's assumption with regard to the latter quantity *drops out as an incidental result in the case of a wire of circular cross-section.*

From equations (6), (7) and (8) we obtain

$$e = f = -\frac{m-n}{2m}g. \tag{9}$$

This result shows that for *any fibre* which is parallel to the axis of the wire, *the ratio of lateral contraction to longitudinal elongation is equal to Poisson's ratio.* Equation (8) only establishes this proposition for the central fibre.

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* Theory of Elasticity, Vol. II, pp. 168–169.

Again, if R be the normal traction at *any point* of the cross-section,

$$\begin{aligned} R &= (m+n)\,g + (m-n)(e+f), \\ &= \frac{(3m-n)\,n}{m}\,g = qg, \end{aligned} \tag{10}$$

by (9), where q is Young's modulus. We have, therefore, proved the following theorem:

When a wire of circular cross-section is twisted as well as bent, the normal traction at any point of a cross-section is equal to the product of Young's modulus and the extension at that point.

6. The values of the two flexural couples are

$$\left.\begin{aligned} G_1 &= \int_0^c \int_0^{2\pi} Rr^2 \sin\theta \, dr\, d\theta, \\ G_2 &= -\int_0^c \int_0^{2\pi} Rr^2 \cos\theta \, dr\, d\theta. \end{aligned}\right\} \tag{11}$$

Now we have stated that in most practical applications the extension of the central axis may safely be neglected; under these circumstances $\sigma_3 = 0$, whence if we substitute the value of R in (11) from (10) and (7), and the values of $\mathfrak{P}$ and $\mathfrak{Q}$ from (4), we shall obtain

$$\left.\begin{aligned} G_1 &= -\tfrac{1}{4}\pi c^4 q \left(\frac{d\mu}{ds} + \frac{\lambda}{\sigma} - \frac{\beta}{\rho}\right), \\ G_2 &= \tfrac{1}{4}\pi c^4 q \left(\frac{d\lambda}{ds} - \frac{\mu}{\sigma}\right), \end{aligned}\right\} \tag{12}$$

which may be written by virtue of (3),

$$\left.\begin{aligned} G_1 &= \tfrac{1}{4}\pi c^4 q \left\{\frac{\beta}{\rho} - \frac{d}{ds}\left(\frac{dv}{ds} + \frac{u}{\sigma}\right) - \frac{1}{\sigma}\left(\frac{du}{ds} + \frac{w}{\rho} - \frac{v}{\sigma}\right)\right\}, \\ G_2 &= \tfrac{1}{4}\pi c^4 q \left\{\frac{d}{ds}\left(\frac{du}{ds} + \frac{w}{\rho} - \frac{v}{\sigma}\right) - \frac{1}{\sigma}\left(\frac{dv}{ds} + \frac{u}{\sigma}\right)\right\}, \end{aligned}\right\} \tag{13}$$

which agree with the expressions for the flexural couples obtained by Mr. Love.

The value of the torsional couple H as shown in my paper is

$$H = \tfrac{1}{2}\pi c^4 n \left(\frac{d\beta}{ds} + \frac{1}{\rho}\,\frac{dv}{ds} + \frac{u}{\rho\sigma}\right). \tag{14}$$

Energy of the Wire.

7. The potential energy of the deformed wire per unit of length is

$$\tfrac{1}{2}\int\!\!\int[(m+n)\Delta^2+n\{a^2+b^2+c^2-4(ef+fg+ge)\}]\,\rho^{-1}(\rho-r\cos\theta)\,r\,dr\,d\theta. \quad (1)$$

By the fundamental hypothesis the strains b and c are to be neglected, since on integration they would lead to terms involving higher powers of the radius of the cross-section than the fourth. The value of the strain a is shown in my paper to be

$$a=\frac{r}{\rho}\left(\frac{d\beta}{d\phi}+\frac{1}{\rho}\,\frac{dv}{d\phi}+\frac{u}{\sigma}\right)=r\left(\frac{d\beta}{ds}+\frac{\mu}{\rho}\right).$$

The values of e, f and g given by (6) and (7) of §5 only include the first power of r, and in order to calculate the potential energy when the central axis is supposed to undergo extension, it would be necessary to proceed to a higher degree of approximation so as to obtain the terms in r^2; for if e contained the term $\mathfrak{M}r^2$, the expression (1) would contain a term $\int\!\!\int\mathfrak{M}\sigma_1 r^3\,dr\,d\theta$ which is proportional to the fourth power of the radius. If, however, the central line is supposed to be inextensible, the terms σ_1, σ_3 are zero, and the expression for the potential energy becomes

$$W=\tfrac{1}{8}\pi c^4\left\{2n\left(\frac{d\beta}{ds}+\frac{\mu}{\rho}\right)^2+q\left(\frac{d\lambda}{ds}-\frac{\mu}{\sigma}\right)^2+q\left(\frac{d\mu}{ds}+\frac{\lambda}{\sigma}-\frac{\beta}{\rho}\right)^2\right\}. \quad (2)$$

Recollecting the values of the couples given by (13) and (14) of §5, and putting A and C for the flexural and torsional rigidities, this may be written

$$W=\tfrac{1}{2}(G_1^2/A+G_2^2/A+H^2/C), \quad (3)$$

a form which is often useful.

The kinetic energy $\mathfrak{T}$ per unit of length is given by the equation

$$\mathfrak{T}=\tfrac{1}{2}\pi hc^2(\dot{u}^2+\dot{v}^2+\dot{w}^2)+\tfrac{1}{4}\pi hc^4\dot{\beta}^2+\tfrac{1}{8}\pi hc^4(\dot{\lambda}^2+\dot{\mu}^2), \quad (4)$$

where h denotes the mass of a unit of length.

8. These formulæ may be verified by means of the variational equation of motion, which thus forms a test of the correctness of the work and of the fundamental hypothesis on which the theory is based.

The equation in question is

$$\delta W+\delta\mathfrak{T}=\delta U+\delta\mathfrak{L},$$

where

$$\delta\mathfrak{T} = \pi hc^2 \int \{\ddot{u}\delta u + \ddot{v}\delta v + \ddot{w}\delta w + \tfrac{1}{2}c^2\ddot{\beta}\delta\beta + \tfrac{1}{4}c^2(\ddot{\lambda}\delta\lambda + \ddot{\mu}\delta\mu)\}\, ds,$$

$$\delta U = \int (X\delta u + Y\delta v + Z\delta w)\, ds,$$

$$\delta\mathfrak{L} = G_2\delta\lambda - G_1\delta\mu + H\delta\beta + N_1\rho\frac{d\delta w}{ds} + N_2\delta v + T\delta w,$$

and if we work out the variation by the ordinary methods of the Calculus of Variations, and take account of the condition of inextensibility, we shall find that (i) we shall reproduce the values of the three couples which we have already obtained; (ii) we shall reproduce the third of equations (1) of §2; (iii) we shall reproduce an equation which is the result of eliminating T between the first and second of (1).

Theory of Finite Deformations.

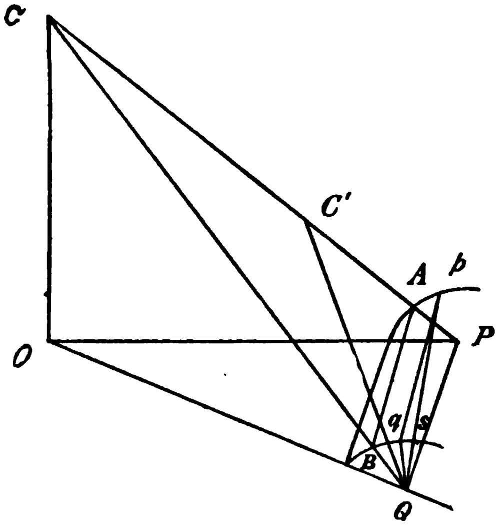
9. When a thin wire is slightly deformed and the central axis undergoes no extension, the expressions for the flexural and torsional couples are given by (13) and (14) of §5; and it is shown in Mr. Love's treatise that the expressions in brackets in these equations are respectively equal to the changes of curvature and twist. It may therefore be anticipated that these couples are proportional to the same geometrical quantities when the deformation is not small; and this we shall now show to be the case.

The method of constructing a theory of finite deformations, which has been adopted by Mr. Love in the second volume of his book, appears to me to be very unsatisfactory and difficult to follow. I do not find the argument on p. 93 at all convincing, and on p. 157 he does not attempt to give any formal proof that these three couples are proportional to the changes of curvature and twist, but dismisses the subject with the perfunctory remark that "as there is some controversy about this result, it may be as well to indicate another method of proof," which occupies a paragraph of about a dozen lines. It was no doubt unfortunate that, owing to the slip which I have corrected in the previous part of this paper, I failed to obtain the correct values of the two flexural couples; but surely the author of what purports to be a classical treatise on Elasticity ought to have cleared up this point and not to have left it in doubt.

10. If we fix our attention on a small element of a finitely deformed wire, whose centre of inertia is P, the displacement of any point Q of the element

will be made up of three distinct parts, viz. (i) a translation of the whole element from one position in space to another, (ii) a rotation of the whole element about some axis through P, (iii) a distortion of the element which produces an alteration in the position of different points of its substance relatively to one another. The first two parts are displacements such as a rigid body might undergo, and cannot therefore produce any strain in the element. The internal strains which the element experiences accordingly depend solely on the third kind of displacement, that is, on the displacement of Q relatively to P.

Let PQ be the central axis of the undeformed element; draw any plane through the tangent at P, let C be the centre of curvature of the element in this plane, and O the centre of principal curvature.



The bending can obviously be effected by means of two opposite couples applied to the ends of the element, whose axes are perpendicular to the plane PCQ, which will be called the *plane of bending*. The effect of the bending will be to change the curvature at P in the plane PCQ, and also to extend or contract all filaments, such as pq, which are parallel to PQ, but the lines pq and PQ will remain sensibly parallel after bending.

The torsion can be effected by applying equal and opposite couples to the ends of the element, whose axes coincide with the tangents at P and Q. The effect of the torsion will be to twist all lines such as pq which lie on the surface of the cylinder $ApqB$ through small angles, so that after torsion they will assume positions slightly inclined to their former ones.

The flexion and torsion will also produce various deformations of a secondary character, but these may be neglected in working out the approximate solution which we shall obtain.

Let ρ, ρ' be the radii of curvature in the plane of bending before and after deformation, ρ_1 the radius of principal curvature; also let $qQO = \theta$, $qQC = \theta'$, $Qq = r$.

Before bending,

$$\frac{pq}{PQ} = \frac{\rho_1 - r\cos\theta}{\rho_1}.$$

The effect of the bending will be to displace the point q through a small space $r\delta\omega\cos\theta'$, where $\delta\omega$ is the rotation due to bending about a line through Q perpendicular to the plane of bending. Now if C' be the centre of curvature in the plane PCQ after bending,

$$\delta\omega = PC'Q - PCQ = PQ\left(\frac{1}{\rho'} - \frac{1}{\rho}\right),$$

whence the displacement of q along pq is

$$r\delta\omega\cos\theta' = PQ\left(\frac{1}{\rho'} - \frac{1}{\rho}\right) r\cos\theta',$$

consequently if σ_3' be the extension,

$$\sigma_3' = -\frac{r\delta\omega\cos\theta'}{pq} = -\left(\frac{1}{\rho'} - \frac{1}{\rho}\right) r\cos\theta', \tag{1}$$

if higher powers of r than the first be neglected.

The effect of the torsion will be to displace the line pq to the position ps, and therefore the above expression for the extension is not rigorously accurate when there is torsion as well as flexion, but the error depends upon the square of the small angle qQs and may be neglected.

If R' be the normal traction perpendicular to the cross-section, we have already proved that *when the deformation is small,*

$$R' = q\sigma_3', \tag{2}$$

where q is Young's modulus; and since our results in the present case must be consistent with those which we have already obtained when the deformation is small, *we shall assume that* (2) *is true when the deformation is finite.* This is the only assumption which it will be necessary to make. Under these circumstances we obtain from (1) and (2),

$$R' = -q\left(\frac{1}{\rho'} - \frac{1}{\rho}\right) r\cos\theta'.$$

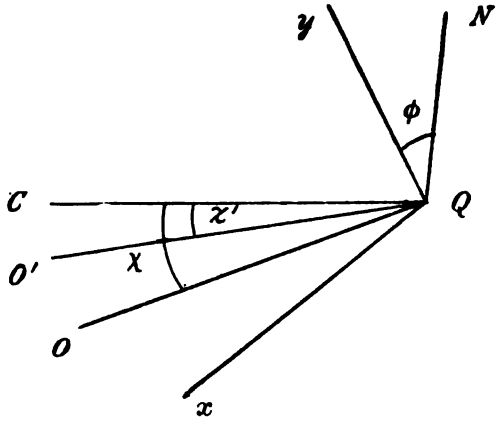
The flexural couple G about the normal through Q to the plane of bending is

$$G = -\int_0^c \int_0^{2\pi} R' r^3 \cos^2 \theta' dr\, d\theta',$$

$$= \tfrac{1}{4}\pi c^4 q \left(\frac{1}{\rho'} - \frac{1}{\rho}\right), \tag{3}$$

and is therefore proportional to the change of curvature in the plane of bending. The flexural couple about QC is obviously zero.

11. We shall now resolve this couple about two arbitrary axes Qx, Qy at right angles to one another in a plane perpendicular to the tangent at Q.



In the figure, QC is the normal to the wire in the plane of bending, QN is the normal to this plane at Q, and C is the centre of curvature in this plane. Let $CQx = NQy = \phi$; then if G_x, G_y be the flexural couples about Qx, Qy, and A the flexural rigidity,

$$G_x = -G \sin\phi = -A\left(\frac{1}{\rho'} - \frac{1}{\rho}\right)\sin\phi, \tag{4}$$

$$G_y = \quad G \cos\phi = \quad A\left(\frac{1}{\rho'} - \frac{1}{\rho}\right)\cos\phi. \tag{5}$$

Let QO, QO' be the principal normals at Q before and after bending; O, O' the centres of principal curvature; also let $CQO = \chi$, $CQO' = \chi'$. Let R_x, R'_x, R_y, R'_y be the radii of curvature before and after bending in planes perpendicular to Qx, Qy, and let ρ_1, ρ'_1 be the radii of principal curvature before and after bending. Then

$$\left.\begin{aligned} \frac{1}{\rho'} &= \frac{1}{\rho_1'}\cos\chi' &, \quad \frac{1}{\rho} &= \frac{1}{\rho_1}\cos\chi &, \\ \frac{1}{R_x'} &= \frac{1}{\rho_1'}\sin(\phi-\chi'), & \frac{1}{R_x} &= \frac{1}{\rho_1}\sin(\phi-\chi), \\ \frac{1}{R_y'} &= \frac{1}{\rho_1'}\cos(\phi-\chi'), & \frac{1}{R_y} &= \frac{1}{\rho_1}\cos(\phi-\chi). \end{aligned}\right\} \tag{6}$$

Since the curvature in the plane through the tangent which is perpendicular to the plane of bending is unchanged,

$$\frac{1}{\rho_1'}\sin\chi' = \frac{1}{\rho_1}\sin\chi. \tag{7}$$

From the first and second of (6) combined with (7) we get

$$\frac{1}{R_x'} = \frac{1}{\rho_1}\sin\phi - \frac{1}{\rho_1}\cos\phi\sin\chi,$$

$$\frac{1}{R_x} = \frac{1}{\rho}\sin\phi - \frac{1}{\rho_1}\cos\phi\sin\chi,$$

whence

$$\frac{1}{R_x'} - \frac{1}{R_x} = \left(\frac{1}{\rho'} - \frac{1}{\rho}\right)\sin\phi,$$

accordingly

$$G_x = -A\left(\frac{1}{R_x'} - \frac{1}{R_x}\right), \tag{8}$$

and in the same way

$$G_y = \quad A\left(\frac{1}{R_y'} - \frac{1}{R_y}\right), \tag{9}$$

where $A = \frac{1}{4}\pi q c^4$ is the flexural rigidity. This shows that the flexural couples about the normals to any two planes at right angles to one another are proportional to the changes of curvature in those planes. The negative sign in (8) is accounted for by the fact that owing to the way in which the quantities are measured G_x is positive when the curvature is diminished.

12. We must now find the torsional couple.

The flexion simply displaces the point q along pq; the torsion produces a displacement along the circular arc to s, so that the line pq assumes the position ps.

Let the angles

$$qps = \psi, \quad qQs = \tau . PQ,$$

then
$$r\tau . PQ = ps . \psi,$$
whence
$$\psi = r\tau . \frac{PQ}{pq} = \frac{\rho r\tau}{\rho - r\cos\theta}:$$

Now ψ is the shearing strain perpendicular to Qq in the plane QqB, whence if H be the torsional couple,

$$\begin{aligned} H &= n\int_0^c\int_0^{2\pi}\psi r^2 dr\, d\theta \\ &= \tfrac{1}{2}\pi c^4 n\tau. \end{aligned} \tag{10}$$

The quantity τ is the change of twist, and is the same quantity which Mr. Love denotes by $\tau' - \tau$.

Potential Energy.

13. Since the work done by a stress is equal to half the product of the stress into the strain produced, it follows that the work done by flexion is

$$\tfrac{1}{2}\int_0^c\int_0^{2\pi} R'\sigma_3' r\, dr\, d\theta = \tfrac{1}{2} q\left(\frac{1}{\rho'} - \frac{1}{\rho}\right)^2 \int_0^c\int_0^{2\pi} r^3 \cos^2\theta\, dr\, d\theta = \tfrac{1}{8}\pi c^4 q\left(\frac{1}{\rho'} - \frac{1}{\rho}\right)^2$$

by (1) and (2).

The work done by torsion is

$$\tfrac{1}{2} n\int_0^c\int_0^{2\pi}\psi^2 r\, dr\, d\theta = \tfrac{1}{4}\pi c^4 n\tau^2.$$

It therefore follows that if W be the potential energy per unit of length,

$$W = \tfrac{1}{4}\pi c^4\left\{\tfrac{1}{2} q\left(\frac{1}{\rho'} - \frac{1}{\rho}\right)^2 + n\tau^2\right\}, \tag{11}$$

which agrees with the results we have already obtained when the deformation is small.

Equilibrium of Naturally Straight Wires.

14. The preceding formulæ can be simplified when the wire is naturally straight. In this case the curvature in every plane through the axis of the wire is zero before deformation; and since the change of curvature in that plane through the tangent to the deformed wire which is perpendicular to the plane of bending is zero after deformation, it follows that the curvature in the above-mentioned plane is also zero after deformation. Hence the plane of bending is the osculating plane of the deformed wire.

From this it follows that

$$G_1 = 0, \quad G_2 = A/\rho,$$

whence, by the fourth of equations (1) of §2,

$$\frac{dH}{ds} = 0,$$

or

$$H = \text{const.},$$

which shows that the torsional couple is constant throughout the length of the wire.

This is a very important proposition.

15. We shall now proceed to integrate the equations of equilibrium of a naturally straight wire.

Since H is constant and $G_1 = 0$, it follows that if ϖ denote the curvature so that $\varpi = 1/\rho$, equations (1) of §2 become

$$\frac{dT}{ds} - N_1\varpi = 0, \tag{1}$$

$$\frac{dN_1}{ds} - \frac{N_2}{\sigma} + T\varpi = 0, \tag{2}$$

$$\frac{dN_2}{ds} + \frac{N_1}{\sigma} = 0, \tag{3}$$

$$\frac{G_2}{\sigma} - H\varpi + N_2 = 0, \tag{4}$$

$$\frac{dG_2}{ds} + N_1 = 0. \tag{5}$$

Since $G_2 = A\varpi$, we obtain from (1) and (5)

$$\frac{dT}{ds} + A\varpi\frac{d\varpi}{ds} = 0,$$

whence

$$T = P - \tfrac{1}{2}A\varpi^2, \tag{6}$$

where P is a constant.

From (4) we get

$$N_2 = \left(H - \frac{A}{\sigma}\right)\varpi, \tag{7}$$

and from (1) and (6)

$$N_1 = -A\frac{d\varpi}{ds}. \tag{8}$$

From (7) and (8) combined with (3) we get

$$H\varpi\frac{d\varpi}{ds} - A\frac{d}{ds}\left(\frac{\varpi^2}{\sigma}\right) = 0,$$

whence

$$\frac{A}{\sigma} = \tfrac{1}{2}H + \frac{Q}{\varpi^2}, \tag{9}$$

where Q is a constant; accordingly by (7)

$$N_2 = \left(\tfrac{1}{2}H - \frac{Q}{\varpi^2}\right)\varpi. \tag{10}$$

To obtain a third integral, substitute the values of T, N_1, N_2 from (6), (8) and (10) in (2) and we get

$$A^2\frac{d^2\varpi}{ds^2} + (\tfrac{1}{4}H^2 - PA)\,\varpi - \frac{Q^2}{\varpi^3} + \tfrac{1}{2}A^2\varpi^3 = 0. \tag{10.A}$$

Integrating we obtain

$$\left(A\varpi\,\frac{d\varpi}{ds}\right)^2 = -\tfrac{1}{4}A^2\varpi^6 + (AP - \tfrac{1}{4}H^2)\,\varpi^4 + R\varpi^2 - Q^2, \tag{11}$$

where R is another constant.

16. From (11) we see that $(d\varpi^2/ds)^2$ is a cubic function of ϖ^2, and therefore ϖ^2 can be expressed in terms of s by means of elliptic functions of the first kind. Let $\tfrac{1}{4}A^2Z$ denote this cubic function; then collecting our results from (6), (9) and (11), we have the following three first integrals of the equations of equilibrium, viz.

$$\left.\begin{aligned} T &= P - \tfrac{1}{2}A\varpi^2, \\ \frac{A}{\sigma} - \tfrac{1}{2}H &= \frac{Q}{\varpi^2} \quad , \\ \frac{d\varpi^2}{ds} &= Z^{\frac{1}{2}} \quad , \end{aligned}\right\} \tag{12}$$

The first of (12) merely determines the tension, but the second leads to important results. If the curve assumed by the wire is a plane curve, $\sigma = \infty$; whence if Q is not zero, ϖ must be constant, and therefore the curve is a circle. If, however, Q is zero, σ is constant, and therefore the curve assumed by the wire is one of constant tortuosity; and if we suppose the curve to be plane, so that $\sigma = \infty$, it follows that H must be zero and the wire devoid of twist. From these results it follows that if a naturally straight wire is twisted as well as bent, the circle is the only plane curve which is a possible figure of equilibrium; but if the wire is bent without being twisted, a family of plane curves exist whose curvature is expressed in terms of the arc by means of the last of (12). The curves are of course the elastica family, whose properties have been discussed by various writers.*

17. We shall now proceed to integrate (11). Writing $\varpi^2 = z$, the equation becomes

$$\left(\frac{dz}{ds}\right)^2 = -z^3 + \left(\frac{4P}{A} - \frac{H^2}{A^2}\right)z^2 + \frac{4R}{A^2}z - \frac{4Q^2}{A^2}. \tag{13}$$

The form of this equation shows that the right-hand side is equivalent to

$$(z+\alpha)(z-\beta)(\gamma-z) = -z^3 + z^2(\beta+\gamma-\alpha) + z(\alpha\beta+\alpha\gamma-\beta\gamma) - \alpha\beta\gamma, \tag{14}$$

and we must now discuss the possible values of α, β, γ.

(i) Let α be real and positive; then if β and γ are real they must both be of the same sign, and this sign must be positive, otherwise $(dz/ds)^2$ would be negative, which is impossible. It also follows that $\gamma > z > \beta$.

If β and γ were complex, we should have

$$(z-\beta)(\gamma-z) = -(z-p-\iota q)(z-p+\iota q),$$

which would make $(dz/ds)^2$ negative.

(ii) Let α be real and negative; then if we suppose β is real and negative we fall back on the previous case with α and β interchanged. But if α is real and negative and β is real and positive, we must have γ real and negative, so that, writing $-\alpha, -\gamma$ for α and γ, the left-hand side of (12) becomes $(\alpha - z)(z-\beta)(z+\gamma)$, which is the first case with α and γ interchanged.

It is also impossible for α to be real and negative and β and γ complex, for this would make the cubic expression complex, since we should have to write $\beta = p + \iota q, \gamma = -(p - \iota q)$.

(iii) Let α be a complex of the form $p + \iota q$, then if γ is real and positive, β must be a complex of the form $p - \iota q$, since the product $\alpha\beta$ must be real and positive; but in this case

$$(z+\alpha)(z-\beta) = (z+p+\iota q)(z-p+\iota q),$$

which is complex.

If γ is real and negative, β must be of the form $-(p-\iota q)$, in which case

$$(z+\alpha)(z-\beta) = (z+p+\iota q)(z+p-\iota q) = (z+p)^2 + q^2,$$

which makes $(dz/ds)^2$ negative.

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* A very complete account of the elastica will be found in Halphen's *Traité des Fonctions Elliptiques*, Vol. II, Ch. V.

(iv) Let α be complex, and β positive; then γ must be of the form $p - \iota q$, so that

$$(z+\alpha)(\gamma - z) = -(z+p+\iota q)(z-p+\iota q),$$

which is complex. If β is negative, we must have $\gamma = -(p-\iota q)$, so that

$$(z+\alpha)(\gamma - z) = -(z+p+\iota q)(z+p-\iota q),$$

which makes $(dz/ds)^2$ negative.

We therefore conclude that the only possible case to consider arises when α, β, γ are all real and positive, and $\gamma > \beta$.

The expression to be integrated now becomes

$$ds = \frac{dz}{\{(z+\alpha)(z-\beta)(\gamma - z)\}^{\frac{1}{2}}}.$$

Let
$$u = (z+\alpha)^{\frac{1}{2}},$$

then
$$ds = \frac{2du}{(u^2 - \alpha - \beta)^{\frac{1}{2}}(\gamma + \alpha - u^2)^{\frac{1}{2}}}.$$

Writing
$$a^2 = \alpha + \gamma, \quad b^2 = \alpha + \beta,$$

we obtain
$$ds = \frac{2du}{(u^2 - b^2)^{\frac{1}{2}}(a^2 - u^2)^{\frac{1}{2}}}.$$

In this write
$$u^2 = a^2\cos^2\phi + b^2\sin^2\phi,$$

and we get
$$\tfrac{1}{2}\,ds = -\frac{d\phi}{\{a^2 - (a^2 - b^2)\sin^2\phi\}^{\frac{1}{2}}},$$

giving
$$u = a\,\mathrm{dn}\,(\tfrac{1}{2}as + \mathfrak{A}), \quad k^2 = (a^2 - b^2)/a^2, \tag{15}$$

where $\mathfrak{A}$ is the constant of integration. We therefore finally obtain

$$\varpi^2 = (\alpha+\gamma)\,\mathrm{dn}^2\{\tfrac{1}{2}(\alpha+\gamma)^{\frac{1}{2}}s + \mathfrak{A}\} - \alpha, \tag{16}$$

$$k^2 = \frac{\gamma - \beta}{\gamma + \alpha}. \tag{17}$$

The constant $\mathfrak{A}$ may be put equal to zero if s be measured from the point where $\varpi = \sqrt{\gamma}$.

18. The greatest value of $\mathrm{dn}\,x$ occurs when $x = 0$ and the least when $x = K$, and since $\gamma > \beta$, it follows that the maxima values of ϖ^2 occur when $\tfrac{1}{2}(\alpha+\gamma)^{\frac{1}{2}}s = 2nK$, in which case ϖ is equal to $\sqrt{\gamma}$, and the minima when $\tfrac{1}{2}(\alpha+\gamma)^{\frac{1}{2}}s = (2n+1)K$ when it is equal to $\sqrt{\beta}$. Hence the curve cannot have any points of inflexion unless $\beta = 0$.

We shall now suppose the wire to form a closed curve, and that there are n maxima and minima values of ϖ; if l be its length, we must have

$$K = \tfrac{1}{2}(\alpha + \gamma)^{\frac{1}{2}}\, l/n.$$

Now the least value of K is $\tfrac{1}{2}\pi$, whence the above equation requires that

$$n < l(\alpha + \gamma)^{\frac{1}{2}}/2\pi,$$

and consequently the number of maxima and minima cannot be greater than the integer which is nearest to $l(\alpha + \gamma)^{\frac{1}{2}}/2\pi$. It also follows that there are no points of inflexion since ϖ^2 can never vanish.

19. The conditions of the problem, as we have already shown, require that α, β, γ should all be positive and that $\gamma > \beta$. It is however possible for α or β to be zero, and we have accordingly two special cases to consider. These particular results may of course be deduced from the general one, but it will be simpler to start from (11).

If α or β is zero, it follows that $Q = 0$, in which case the curve assumed by the wire is one of constant tortuosity; under these circumstances ϖ^2 cuts out from both sides of (11) and the equation becomes

$$4\left(\frac{d\varpi}{ds}\right)^2 = \frac{4R}{A^2} - \left(\frac{H^2}{A^2} - \frac{4P}{A}\right)\varpi^2 - \varpi^4. \tag{18}$$

Since $d\varpi/ds$ is essentially a real quantity, it follows that if $H^2/A > 4P$, R cannot be negative, and consequently if R is negative H^2/A must be less than $4P$, and we have therefore two cases to consider.

20. Case I. Let R be positive, and write

$$\left.\begin{aligned} a^2b^2 &= \frac{4R}{A^2}, \\ a^2 - b^2 &= \frac{H^2}{A^2} - \frac{4P}{A}, \end{aligned}\right\} \tag{19}$$

then (18) becomes

$$2\frac{d\varpi}{ds} = (a^2 + \varpi^2)^{\frac{1}{2}}(b^2 - \varpi^2)^{\frac{1}{2}},$$

which shows that this corresponds to the case of $\beta = 0$. Putting $\varpi = b\cos\phi$, we get

$$-2\frac{d\phi}{ds} = (a^2 + b^2 - b^2\sin^2\phi)^{\frac{1}{2}},$$

whence
$$\varpi = b \operatorname{cn} \tfrac{1}{2}(a^2 + b^2)^{\frac{1}{2}} s, \qquad (20)$$
$$k = \frac{b}{(a^2 + b^2)^{\frac{1}{2}}}, \qquad (21)$$

the constant being chosen so that $\varpi = b$ when $s = 0$.

Since ϖ vanishes and changes sign when
$$\tfrac{1}{2}(a^2 + b^2)^{\frac{1}{2}} s = (2n + 1) K,$$
it follows that the curve has points of inflexion.

21. Case II. Let R be negative. We must now write
$$\left.\begin{aligned} a^2b^2 &= -\frac{4R}{A^2}, \\ a^2 + b^2 &= \frac{4P}{A} - \frac{H^2}{A^2}, \end{aligned}\right\} \qquad (22)$$
and (18) becomes
$$2\frac{d\varpi}{ds} = (a^2 - \varpi^2)^{\frac{1}{2}}(\varpi^2 - b^2)^{\frac{1}{2}},$$
which corresponds to $\alpha = 0$ in the general case. The integral of this is
$$\left.\begin{aligned} \varpi &= a \operatorname{dn} \tfrac{1}{2} as, \\ k &= (a^2 - b^2)^{\frac{1}{2}}/a, \end{aligned}\right\} \qquad (23)$$
and since ϖ can never vanish and change sign, there are no points of inflexion.

Stability of a Deformed Elastic Wire.

22. The stability of a deformed elastic wire may be investigated by three methods, which we shall proceed to explain.

23. The first method consists in supposing the wire to perform small oscillations about its configuration of equilibrium and finding their periods; the condition of stability is that the roots of the period equation should be real. This method possesses advantages when the periods are of acoustical interest; its chief defect is that it is somewhat indirect, and often leads to rather long and complicated expressions.

24. The second method, which has been employed by Prof. Greenhill* in considering the stability of a column under thrust and twist, depends upon some-

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* Proc. Inst. Mech. Engineers, 1883.

what refined considerations, and will be best illustrated by considering a special problem.

Let a naturally straight wire be bent and twisted and the ends joined together. It is easy to show, and will afterwards be proved, that a circle is a possible figure of equilibrium. Let us now assume that the circular form is stable when the torsional couple H is less than H_0, where H_0 is the quantity whose value we wish to determine. If H is very slightly greater than H_0, a figure of equilibrium will exist in which the wire assumes the form of a sinuous curve which differs very slightly from a circle, and will be derived therefrom by supposing the wire to undergo small displacements u, v, w and β. We must therefore solve the equations of equilibrium on the supposition that the sinuous form is a possible one when the torsional couple has an arbitrary value H. Since u, v, w and β must be periodic with respect to the vectorial angle ϕ, each of these quantities must be proportional to $\varepsilon^{\iota s\phi}$, where s is any integer greater than unity, since $s=0$, $s=1$ correspond to rigid body displacements which can produce no alteration in the state of strain; and we shall thus be led to an equation of the form

$$H = F(s).$$

When H is less than the minimum value of $F(s)$, if such exist, it will be impossible to satisfy this equation, and consequently equilibrium in the sinuous form cannot exist, from which it follows that equilibrium in the circular form is stable. But when H is slightly greater than the minimum value of $F(s)$, equilibrium in the sinuous form is possible, and the precise form of the curve can be determined by means of a Fourier's series. If H_0 denote this minimum value, the condition of stability is that

$$H < H_0.$$

25. The third method is the energy method, and the condition of stability is that the potential energy in the configuration of equilibrium should be a minimum. If G be the resultant flexural couple, A and C the flexural and torsional rigidities,* the value of the potential energy per unit of length is

$$W = \tfrac{1}{2}(G^2/A + H^2/C).$$

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* The cross-section of the wire is supposed to be circular. There does not appear to be much advantage in taking into account any deviation from circularity in wires ordinarily met with. When the wire is a flat one, resembling a clock-spring, the theory of thin plates is more applicable.

Let $W+W'$, $G+G'$, $H+H'$ be the values of the potential energy and the flexural and torsional couples in any slightly displaced configuration; then

$$W' = \tfrac{1}{2}(G'^2/A + H'^2/C + 2GG'/A + 2HH'/C).$$

Now G' and H' are proportional to the small changes of curvature and twist which occur in passing from the equilibrium to the disturbed configuration, and these quantities can be found in terms of the displacements by means of the formulæ given by Mr. Love on p. 168 of his book; but in order to apply the energy method it is essential that we should know the correct expression for the potential energy to the *second* order of small quantities, whereas Mr. Love's formulæ only give the correct values of the product terms in W' to the first order. This method would therefore require us to calculate the changes of curvature and twist to the second order of small quantities; and this has not yet been done.

Stability of a Straight Wire subjected to Thrust.

26. The stability of a straight wire which is subjected to thrust is discussed in (amongst other places) Mr. Love's Treatise on Elasticity; but as his investigation does not bring out at all clearly the precise nature of the terminal conditions, I shall consider the subject afresh.

As there is no torsional couple, it will be sufficient to treat the problem as one of two dimensions. From the argument in §25, it follows that when the pressure or thrust at the extremities of the wire is sufficiently great, the wire will begin to bend and to assume the form of a curve of the elastica family, and we have to find the value of the thrust which is just sufficient to produce this state of things.

The equations of equilibrium of the deformed wire are

$$\frac{dT}{ds} - N\varpi = 0, \tag{1}$$

$$\frac{dN}{ds} + T\varpi = 0, \tag{2}$$

$$A\frac{d\varpi}{ds} + N = 0, \tag{3}$$

since $G = A\varpi$. From (1) and (3) we get

$$T = -P - \tfrac{1}{2}A\varpi^2,$$

as before; and since the curvature is a very small quantity, we may in (2) put $T = -P$, where P is the thrust applied to the ends of the wire. Whence by (2) and (3),

$$A\frac{d\varpi^2}{ds^2} + P\varpi = 0,$$

the integral of which is

$$\varpi = C\cos\mu s + D\sin\mu s,\quad \mu^2 = P/A. \tag{4}$$

27. Case I. Let the lower end A of the wire be firmly clamped, whilst the upper end B is pressed vertically downwards by a force P, but is otherwise free; also let l be the length of the wire, and let the arc s be measured from A.

At the end B, G and therefore ϖ are zero; whence

$$C\cos\mu l + D\sin\mu l = 0. \tag{5}$$

Also by considering the equilibrium of the whole wire, it follows that $N = 0$ at A, whence by (3) $d\varpi/ds = 0$ when $s = 0$, accordingly $D = 0$. This requires that $\cos\mu l = 0$, whence $\mu l = (2n+1)\frac{1}{2}\pi$, or

$$P = \tfrac{1}{4}\pi^2(2n+1)^2 A/l^2. \tag{6}$$

The least value of the right-hand side of (6), which occurs when $n = 0$, gives the thrust P which must be applied to the upper end of the wire to produce an infinitesimally small deflection; if, therefore, the thrust is less than this quantity, no deflection will take place and the wire will remain straight. Whence the condition of stability is that

$$P < \tfrac{1}{4}\pi^2 A/l^2. \tag{7}$$

28. Case II. Let the wire be pressed between two parallel planes which are perpendicular to its undisplaced position. If the planes were perfectly hard, smooth and rigid (a condition which can only be approximately realized in nature), the ends of the wire would tend to slip on the slightest pressure being applied; we shall therefore suppose that the ends are in contact with mechanical appliances which will prevent any such slipping taking place, but are otherwise free.

Under these circumstances the terminal conditions are $\varpi = 0$ when $s = 0$ and $s = l$. Whence by (4)

$$C = 0,\quad \sin\mu l = 0,\quad \mu l = \pi,$$

and the condition of stability is that

$$P < \pi^2 A/l^2. \tag{8}$$

29. Case III. Let both ends of the wire be clamped. The terminal conditions require that the values of N and ϖ at the two ends should be equal to one another. Consequently

$$\begin{aligned} D(1-\cos\mu l) &= -\,C\sin\mu l, \\ C(1-\cos\mu l) &= \quad D\sin\mu l. \end{aligned}$$

Eliminating C and D we obtain

$$\sin^2 \tfrac{1}{2}\mu l = 0, \quad \tfrac{1}{2} l = \pi,$$

and the condition of stability is

$$P < 4\pi^2 A/l^2. \tag{9}$$

The value of ϖ may now be written

$$\varpi = C\cos 2\pi s/l,$$

which shows that there are two points of inflexion, which occur when $s = \tfrac{1}{4}l$ and $s = \tfrac{3}{4}l$.

The first case corresponds to a column or pillar whose lower end is cemented into a bed of concrete, whilst the upper end supports a building which simply rests upon but is not fastened to the pillar; and we see that in this case the weight required to cause the pillar to collapse is less than in the other two cases. The second case corresponds to a pillar or rod both of whose ends rest on bearings to which they are not cemented. The third case corresponds to a pillar whose ends are respectively cemented to the foundations and the building supported. In the third case, the force required to cause the pillar to collapse is four times greater than in the second and sixteen times greater than in the first case.

Stability of a Straight Wire under Thrust and Twist.

30. We shall now suppose that a torsional couple is applied to the ends of the wire as well as a thrust, and shall investigate the conditions of stability.

In Case I, ϖ vanishes at one end because there is no flexural couple there; in Case II it vanishes at both ends; whilst in Case III the tangents at both extremities are parallel, and consequently there are at least two points of inflexion. Now from (16) of §17, and (23) of §21, it follows that ϖ can never vanish unless the constant β which occurs in these equations is zero; accordingly the constant Q which appears in (12) of §16 must be zero.

The curvature is given by (10.A) of §15, in which Q must be put equal to zero and P to $-P$; also the term $\frac{1}{2}A^2\varpi^3$ being of the third order must be neglected and the equation becomes

$$\frac{d^2\varpi}{ds^2} + \left(\frac{H^2}{4A^2} + \frac{P}{A}\right)\varpi = 0, \tag{1}$$

where P now denotes the thrust; also by (8) and (10) of §15,

$$N_1 = -A\frac{d\varpi}{ds}, \tag{2}$$

$$N_2 = \tfrac{1}{2}H\varpi. \tag{3}$$

We have therefore

$$\varpi = C\cos\mu s + D\sin\mu s, \tag{4}$$

where

$$\mu^2 = \frac{H^2}{4A^2} + \frac{P}{A}. \tag{5}$$

Case I. At the end B, $\varpi = 0$, whence

$$C\cos\mu l + D\sin\mu l = 0, \tag{6}$$

whilst at the end A the shearing stress N_1, which is along the principal normal, must also vanish, which gives $D = 0$. Consequently the condition of stability is

$$\frac{H^2}{4A^2} + \frac{P}{A} < \frac{\pi^2}{4l^2}. \tag{7}$$

Case II. Here $\varpi = 0$ when $s = 0$ and $s = l$, when the condition will be found to be

$$\frac{H^2}{4A^2} + \frac{P}{A} < \frac{\pi^2}{l^2}. \tag{8}$$

Case III. In this case symmetry requires that the values of ϖ and N_1 should be equal when $s = 0$ and $s = l$. This leads to the condition

$$\frac{H^2}{4A^2} + \frac{P}{A} < \frac{4\pi^2}{l^2}. \tag{9}$$

All these results agree with our former ones, as can be seen by putting $H = 0$.

Equilibrium and Stability of a Naturally Straight Wire deformed into a Helix.

31. It has been well known for many years that a helix is a possible figure of equilibrium for a naturally straight wire which is twisted as well as bent.

This result may easily be deduced by means of the general equations of equilibrium (1) of §2. In the helix

$$\frac{1}{\rho}=\frac{\cos^2\alpha}{a},\quad \frac{1}{\sigma}=\frac{\sin\alpha\cos\alpha}{a},\tag{1}$$

where α is the pitch and a is the radius of the cylinder on which the helix is traced; also in the helix the principal normal is the normal to the cylinder.

Since the natural form of the wire is straight,

$$H=\text{const.}\quad G_1=0,\quad G_2=A/\rho;\tag{2}$$

also none of the quantities can be functions of s, whence it follows from the general equations that

$$N_1=0,\quad T=\frac{H}{\sigma}-\frac{A}{\sigma^2},\quad N_2=\frac{H}{\rho}-\frac{A}{\rho\sigma}.\tag{3}$$

These equations combined with (1) give

$$\left.\begin{aligned}T&=\frac{H\sin\alpha\cos\alpha}{a}-\frac{A\sin^2\alpha\cos^2\alpha}{a^2},\\ N_2&=\frac{H\cos^2\alpha}{a}-\frac{A\sin\alpha\cos^3\alpha}{a^2},\\ G_2&=\frac{A\cos^2\alpha}{a},\end{aligned}\right\}\tag{4}$$

whence

$$\left.\begin{aligned}T\cos\alpha-N_2\sin\alpha&=0,\\ T\sin\alpha+N_2\cos\alpha&=\frac{H}{a}\cos\alpha-\frac{A\sin\alpha\cos^2\alpha}{a^2}.\end{aligned}\right\}\tag{5}$$

Equations (5) show that the resultant force F which must be applied to the ends of the wire must be parallel to the axis of the cylinder on which the helix is traced, and that its magnitude is

$$F=\frac{H}{a}\cos\alpha-\frac{A}{a^2}\sin\alpha\cos^2\alpha.\tag{6}$$

The resultant couple $\mathfrak{G}$ is

$$\mathfrak{G}^2=H^2+\frac{A^2}{a^2}\cos^4\alpha.\tag{7}$$

The resultant force and couple are therefore to a certain extent arbitrary, since both contain the torsional couple H, the only limitation on whose value is that it must not be large enough to break the wire or to produce a permanent set. We have therefore two special cases to consider.

32. Case I. Let $H=0$; then the terminal stresses consist of a pushing force or thrust P, whose value is

$$P=A/a^2.\sin\alpha\cos^2\alpha,$$

together with a flexural couple G_2, whose value is $A/a.\cos^2\alpha$. The pitch of the helix is $\sin^{-1}(Pa/G_2)$, from which we see that in order that equilibrium may be possible Pa must not be greater than G_2.

33. Case II. Let $F=0$, then

$$H=\frac{A}{a}\sin\alpha\cos\alpha=\frac{A}{\sigma}, \tag{8}$$

whilst $\mathfrak{G}=A/a.\cos\alpha$. The torsional couple is therefore proportional to the tortuosity; also since

$$\begin{aligned}H\cos\alpha-G_2\sin\alpha&=0,\\ H\sin\alpha+G_2\cos\alpha&=A/a.\cos\alpha,\end{aligned}$$

it follows that the terminal stress consists of a couple whose axis is parallel to the axis of the cylinder on which the helix is traced, and whose magnitude is $A/a.\cos\alpha$.

34. We shall now suppose that the wire is bent and twisted into a helix and the ends firmly clamped; and we shall investigate the condition that the helical form may be stable.

Let the wire be slightly displaced from its equilibrium configuration, and let $\rho^{-1}+p$ be its curvature. Substituting this value of ϖ in (10.A) of §15, we obtain

$$\frac{d^2p}{ds^2}+\left(\frac{H^2}{4A^2}-\frac{P}{A}+\frac{3Q^2\rho^4}{A^2}+\frac{3}{2\rho^2}\right)p=0, \tag{9}$$

which determines the small change of curvature.

By (3), (6) and (9) of the present article, we obtain

$$P=\frac{H}{\sigma}-\frac{A}{\sigma^2}+\frac{A}{2\rho^2},$$

$$Q=\left(\frac{A}{\sigma}-\tfrac{1}{2}H\right)\frac{1}{\rho^2},$$

consequently (9) becomes

$$\frac{d^2p}{ds^2} + \left\{\left(H - \frac{2A}{\sigma}\right)^2 + \frac{A^2}{\rho^2}\right\}\frac{p}{A^2} = 0, \tag{10}$$

the solution of which is

$$p = C\cos\mu s + D\sin\mu s, \tag{11}$$

where

$$A^2\mu^2 = \left(H - \frac{2A}{\sigma}\right)^2 + \frac{A^2}{\rho^2}. \tag{12}$$

Let R, Θ, Z be the stresses along and perpendicular to the radius and parallel to the axis of the cylinder upon which the helix is traced; then

$$\left.\begin{aligned} N_1 &= -R, \\ T\cos\alpha - N_2\sin\alpha &= \Theta, \\ T\sin\alpha + N_2\cos\alpha &= Z. \end{aligned}\right\} \tag{13}$$

Since these equations are true in the case of the helical and the disturbed configuration, they will also be true when the variations of the stresses are substituted for their original values, in which case we have from (6), (8) and (10) of §15,

$$\left.\begin{aligned} T &= -\frac{Ap}{\rho}, \\ N_1 &= -A\frac{dp}{ds}, \\ N_2 &= (\tfrac{1}{2}H + Q\rho^2)p. \end{aligned}\right\} \tag{14}$$

Let us now suppose that the two ends of the wire lie on the same generator of the cylinder, so that the wire forms an even number of complete convolutions. From (13) and (14) the terminal conditions give

$$\left(\frac{dp}{ds}\right)_0 = \left(\frac{dp}{ds}\right)_l,$$

$$p_0 = p_l,$$

which by (11) become

$$-D(1 - \cos\mu l) = C\sin\mu l,$$

$$C(1 - \cos\mu l) = D\sin\mu l,$$

whence, eliminating C and D, we get

$$\sin^2\tfrac{1}{2}\mu l = 0,$$

or

$$\mu l = 2\pi.$$

Using the value of μ given by (12), we get

$$\left(H-\frac{2A}{\sigma}\right)^2+\frac{A^2}{\rho^2}=\frac{4\pi^2A^2}{l^2}. \tag{15}$$

Now (6) may be written

$$F\sin\alpha=\frac{H}{\sigma}-\frac{A}{\sigma^2}, \tag{16}$$

accordingly by (15) the condition of stability becomes

$$\frac{\mathfrak{G}^2}{A^2}-\frac{4F\sin\alpha}{a}<\frac{4\pi^2}{l^2}. \tag{17}$$

Since the ends of the wire are supposed to lie on the same generator of the cylinder, there must be m convolutions; whence the pitch of the helix is determined by the equation

$$2m\pi a\sec\alpha=l, \tag{18}$$

where m is an integer.

From (17) and (1) the condition may be written

$$\left(H-\frac{2A}{\sigma}\right)^2<\frac{4A^2\pi^2}{l^2}(1-m^2\cos^2\alpha),$$

which is impossible unless $\sec\alpha>m$.

When there is only one convolution $m=1$ and the condition becomes

$$H<\frac{6\pi A}{l}\sin\alpha, \qquad \alpha>0.$$

35. We shall now consider the two special cases.

In Case I the helix is held in equilibrium by a flexural couple and a thrust, and the condition (17) becomes

$$\cos^2\alpha\,(1+3\sin^2\alpha)<4\pi^2a^2/l^2<\cos^2\alpha/m^2,$$

or

$$1+3\sin^2\alpha<m^{-2},$$

which is impossible, and therefore the equilibrium is unstable.

In Case II the wire is held in equilibrium by a couple whose value is $A/a.\cos\alpha$, and the condition becomes $m<1$, which is impossible.

It therefore follows that in the two special cases the wire is unstable when it makes one complete convolution. When the wire does not make a complete convolution, the terminal conditions, and consequently the conditions of stability, will be represented by a different set of equations; but the investigation of the various cases which arise may be left to the reader.

Equilibrium and Stability of Circular Wire.

36. We shall now consider a problem concerning the stability of a naturally curved wire which covers a good many special cases.

A wire whose natural form is a tortuous curve is first unbent; secondly, the wire is twisted, and thirdly, the ends are joined together; it is required to find the condition that the circle is a possible figure of equilibrium, and that the circular form may be stable.

When the circle is a figure of equilibrium none of the quantities can be functions of s; we therefore obtain from (1) of §2,

$$\left.\begin{array}{lll} T=0\quad, & N_1=0\quad, & G_1=0\quad, \\ H=\text{const.} & G_2=\text{const.} & N_2=H/a. \end{array}\right\} \tag{1}$$

The constancy of G_2 and H requires that the changes of curvature and twist which occur in passing from the natural to the circular form shall be constant quantities. These conditions will be satisfied if the natural form of the wire is a helix, which includes as a particular case a circular coil of *fine* wire, the radius of whose cross-section is small in comparison with the mean radius of the coil.

37. To investigate the stability we shall employ the second method. In the circular and the sinuous forms respectively let $H=C\tau$, and $H+H'=C\tau'$ be the torsional couples; then $H'=C(\tau'-\tau)$ where $\tau'-\tau$ is the small change of twist which occurs in passing from the circular to the sinuous form. This is a small quantity which can be expressed in terms of the four displacements u, v, w and β which the wire experiences in passing from one form to the other, whence by the third of equations (38) on p. 168 of Mr. Love's book,

$$H'=\frac{C}{a}\left(\frac{d\beta}{d\phi}+\frac{1}{a}\frac{dv}{d\phi}\right). \tag{2}$$

Let G_1', G_2+G_2' be the flexural couples when the wire is sinuous; then from the same equations it follows that

$$\left.\begin{array}{l} G_1'=\dfrac{A}{a}\left(\beta-\dfrac{1}{a}\dfrac{d^2v}{d\phi^2}\right), \\ G_2'=\dfrac{A}{a^2}\left(\dfrac{d^2u}{d\phi^2}+u\right)\qquad, \end{array}\right\} \tag{3}$$

since the wire is supposed to be inextensible.

The radius of torsion of the sinuous curve is given by equation (23) on p. 163 of Love's book, and in the present case is

$$\frac{1}{\sigma'} = \frac{1}{a^2}\left(\frac{d^3v}{d\phi^3} + \frac{dv}{d\phi}\right). \tag{4}$$

The equations of equilibrium when the wire is slightly sinuous are now

$$\left.\begin{aligned}
\frac{dT}{d\phi} - N_1 &= 0,\\
\frac{dN_1}{d\phi} - \frac{H}{a^2}\left(\frac{d^3v}{d\phi^3} + \frac{dv}{d\phi}\right) + T &= 0,\\
\frac{dN_2'}{d\phi} \qquad &= 0,\\
\frac{dH'}{d\phi} - G_1' &= 0,\\
\frac{dG_1'}{d\phi} - \frac{G_2}{a}\left(\frac{d^3v}{d\phi^3} + \frac{dv}{d\phi}\right) + \frac{H}{a}\left(\frac{d^2u}{d\phi^2} + u\right) + H' - N_2' a &= 0,\\
\frac{1}{a}\,\frac{dG_2'}{d\phi} + N_1 &= 0.
\end{aligned}\right\} \tag{5}$$

Let $D = d/d\phi$; then by (2) and (3) the fourth of (5) becomes

$$a\,(CD^2 - A)\,\beta + (A + C)\,D^2 v = 0. \tag{6}$$

Differentiating the fifth and taking into account the fourth of (16) and also (2) and (3) we get

$$(A + C)\,D^2\beta + a^{-1}\{C - AD^2 - aG_2\,(D^2 + 1)\}\,D^2 v + H\,(D^2 + 1)\,Du = 0. \tag{7}$$

Eliminating T and N_1 between the first, second and sixth, we get

$$Ha\,(D^2 + 1)\,D^2 v + A\,(D^2 + 1)^2\,Du = 0. \tag{8}$$

Since the circle is *complete*, all the quantities must be functions of $\varepsilon^{\iota s\phi}$, where s is any integer greater than unity, since $s = 0$ and $s = 1$ correspond to rigid body displacements which can produce no alteration in the state of strain. Our equations accordingly become

$$\left.\begin{aligned}
&a\,(Cs^2 + A)\,\beta + (A + C)\,s^2 v = 0,\\
&(A + C)\,s\beta + a^{-1}\{C + As^2 + aG_2\,(s^2 - 1)\}sv + \iota H\,(s^2 - 1)\,u = 0,\\
&Hasv + \iota A\,(s^2 - 1)\,u = 0,
\end{aligned}\right\} \tag{9}$$

Eliminating u, v and β we get

$$H^2a^2 = \left\{ G_2 a + \frac{AC(s^2-1)}{A+Cs^2} \right\} A(s^2-1). \tag{10}$$

If, therefore,

$$H^2a^2 < \left\{ G_2 a + \frac{AC(s^2-1)}{A+Cs^2} \right\} A(s^2-1), \tag{11}$$

it will be impossible to satisfy the conditions of equilibrium, consequently (11) is the condition of stability.

If the curvature is increased by deformation, G_2 is a positive quantity, and the least value of the right-hand side of (11) occurs when $s = 2$; under these circumstances the condition becomes

$$H^2a^2 < 3A \left\{ G_2 a + \frac{3AC}{A+4C} \right\}. \tag{12}$$

But if, on the other hand, the curvature is diminished by deformation, G_2 will be negative, and the inequality (12) involves the subsidiary condition that

$$G_2 a + \frac{3AC}{A+4C} \tag{13}$$

should be positive.

38. Before discussing the general condition, it will be desirable to consider the subsidiary condition (13). Let ρ be the radius of curvature of the undeformed wire, and let $\rho < a$; then

$$-G_2 a = A\left(\frac{a}{\rho} - 1\right),$$

and the condition becomes

$$\frac{a}{\rho} - 1 < \frac{3C}{A+4C}.$$

For metal wires $q/n = \frac{5}{2}$ about, so that $A/C = \frac{5}{4}$, and the last equation becomes

$$a < \tfrac{11}{7}\rho. \tag{14}$$

It therefore follows that the circular form will be unstable if its radius is greater than $\frac{11}{7}$ths of the radius of principal curvature of the undeformed wire.

When the natural form of the wire consists of a circular coil which is unrolled and the ends joined together, the preceding result shows that the circular form will be unstable when the length of the coil is greater than about one and a half complete convolutions. We shall presently consider what will happen when this length is exceeded.

39. When the natural form of the wire consists of a *complete* circle of radius a, which is cut and then twisted and its ends joined together, $G_2 = 0$ since the twist produces no change of curvature. Under these circumstances the condition (12) becomes

$$Ha < 3A\sqrt{\frac{C}{A + 4C}}.$$

Assuming that $A/C = \frac{4}{5}$, this condition requires that the total twist should not be greater than $\pi \times 3.27$; that is, about six and a half right angles.

Let the natural form of the wire be a helix of pitch α, and let $2\pi l$ be the length of a complete convolution. Then $l \cos\alpha$ is the radius of the cylinder upon which the helix is traced, and $l \sec\alpha$ is its radius of curvature; whence the subsidiary condition (13) becomes

$$a \cos\alpha < \tfrac{11}{7} l;$$

that is to say, the projection of the length of the wire upon a circular section of the cylinder must not be greater than $\frac{11}{7}$ths of a complete convolution.

When the above condition is satisfied, the circular form will be stable when a torsional couple is applied to the ends of the wire before they are soldered together, provided the twist does not exceed a certain magnitude which is determined by (12).

40. We shall now consider the period equation when a complete circular wire is performing small oscillations about its configuration of stable equilibrium. The method employed is precisely similar to the investigation given on p. 121 of my paper on wires,* and equation (50) on p. 122 is a particular case of the more general result which we shall proceed to consider. I find that the periods are given by the following cubic equation:

$$\begin{aligned}
2\pi^2 c^4 a^6 \Big[& a^2 (1 + \tfrac{1}{4}\lambda^2 s^2) h^2 p^4 \\
& - \tfrac{1}{2}\{q + 2ns^2 + \tfrac{1}{4}\lambda^2 s^2 (q + 2qs^2 + 4n + 2ns^2) + \tfrac{1}{4}\lambda^2 s^2 (s^2 - 1) qR\} hp^2 \\
& + \frac{\lambda^2 q s^2 (s^2 - 1)}{4a^2} \{n (s^2 - 1) + \tfrac{1}{2} (q + 2ns^2) R\} \Big] \\
& \qquad \times \Big[\{s^2 + 1 + \tfrac{1}{4}\lambda^2 (s^2 - 1)^2\} hp^2 - \frac{\lambda^2 q s^2 (s^2 - 1)^2}{4a^2} \Big] \\
& + H^2 s^4 (s^2 - 1)^2 (q + 2ns^2 - 2a^2 hp^2) = 0. \qquad (15)
\end{aligned}$$

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* Proc. Lond. Math. Soc., Vol. XXIII.

In this equation c is the radius of the cross-section $\lambda = c/a$, q and n are Young's modulus and the rigidity, h the density, p the period, and $R = 1 - a/\rho$, where ρ is the radius of curvature of the natural form.

Omitting superfluous positive factors, the term independent of p is

$$H^2a^2 - \tfrac{1}{8}\pi^2q^2c^8(s^2 - 1)\left\{\frac{n(s^2 - 1)}{q + 2ns^2} + \tfrac{1}{2}R\right\},$$

and the condition that one of the roots of the cubic should be real and positive is that this quantity should be negative. This condition is easily seen to be equivalent to (11).

To investigate the conditions that the remaining roots should be real and positive would be a somewhat troublesome operation; but there can be little doubt that the conditions of stability already given are sufficient to insure that this should be the case.

41. We shall now consider the case in which there is no twist. Under these circumstances the cubic splits up into two factors, the second of which gives the periods of the vibrations of a Hoppe's ring, whilst the first factor leads to an equation equivalent to (50) of my former paper, with which it becomes identical when $R = 0$, as was the case in the problem there considered. We therefore see that the vibrations consist of two distinct types, viz. flexural vibrations in the plane of the ring, and vibrations which involve torsion and flexion perpendicular to this plane. The periods of the purely flexural vibrations are always real, and consequently the ring is stable for displacements in its own plane; but if R is a negative quantity whose numerical value is greater than the least value of $2n(s^2 - 1)/(q + 2ns^2)$, the absolute term of the first factor will be negative and the motion will be unstable. This leads to the subsidiary condition (13).

From these results we see that when the circular form becomes unstable the ring will not collapse like a boiler flue, but will assume the form of a bent and twisted tortuous curve. They also to a certain extent show what the form of this curve will be. Assuming that $A/C = \frac{5}{4}$, it follows that the value of

$$\frac{C(s^2 - 1)}{A + Cs^2}$$

when $s = 2$ is $\frac{4}{7}$, and its value when $s = \infty$ is unity. If therefore G_2a/A has a negative value whose numerical value lies between $\frac{4}{7}$ and 1, a sinuous figure differing slightly from a circle will be possible; but if this numerical value is considerably greater than unity, a sinuous form will be impossible and the unstable circle will make a sudden jump, and will assume the form of some entirely different curve or may even turn itself inside out.

42. A great many other special problems can be solved by the above methods; but when a wire whose natural form is a straight line is twisted and the ends soldered together, the condition of stability cannot be obtained by means of the general formula (11). The problem is one of those special cases which so frequently occur in mathematics in which a formula of apparent generality fails to give a correct result in some particular instance and a procedure of a special kind has to be resorted to. When the wire is naturally straight H' is constant and G_1' is zero, so that the fourth of (5) disappears as well as the term $dG_1'/d\phi$ in the fifth equation. Under these circumstances the first of (9) disappears whilst the second becomes, since $G_2 a = A$,

$$Hau - \iota Asv = 0.$$

The third of (9) remains unaltered, so that we get

$$H'^2 a^2 = A^2 (s^2 - 1),$$

which gives

$$\tau = \frac{q\sqrt{3}}{2na}$$

as the condition of stability.

This result appears to have been first given by Mr. Michell,[*] who obtained it by supposing the wire to perform small oscillations. Assuming that $q/n = \frac{5}{2}$, he found that the total twist must be less than $2\pi \times 2.16$, that is, less than eight and a half right angles.

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[*] Mess. Math., Vol. XIX, p. 184.

Investigations in the Lunar Theory.

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This paper is an outline of a plan for the development of the expressions which represent the coordinates of the Moon, together with certain theorems connected with the infinite determinants which determine the motions of Perigee and Node, and, in addition, some results concerning the constant part of the expression which gives the Parallax of the Moon.

It has been pointed out that the algebraical expressions of the coordinates show slow convergence chiefly when the coefficients of the periodic terms are arranged in powers of the ratio of the mean motions of the Sun and Moon, and that when we use the numerical value of this ratio from the start, keeping the other constants arbitrary, slow convergence in the series arranged in powers and products of the other constants, is unusual; also, the observed value of the ratio of the mean motions is determined with an accuracy far surpassing that with which the other constants are known. These reasons have seemed to point towards a semi-algebraical development, in which the ratio of the mean motions is given its numerical value while the other constants are left arbitrary.

The developments given below are built up on this basis. The orbit of Dr. Hill, which depends on m only, is, therefore, a numerical one, and is used as a first approximation or "intermediate" orbit.

But there are several methods by which we might proceed with the development from this point. It is possible to use polar or rectangular coordinates, and the equations of motion in each case can be given several different forms. Of the forms which the equations may take when rectangular coordinates are used, there are two which seem specially adapted for continued approximation. Of these one set are of the second degree and second order when we neglect the Parallax of the Sun and the Latitude of the Moon. When these quantities are included, two equations still keep a form in which the principal terms are of the same kind, while the third equation is a sufficiently simple one. These are the equations (8) below. Some of the elliptic and parallactic terms have been determined from these equations.

Another method is to use the equations of motion in their original form. This requires the expansion of such expressions as $(x_0 + \delta x)/(r_0 + \delta r)^3$, where x_0, r_0 represent the values of x, r in the intermediate orbit. These being trigonometrical or exponential series, entail a certain amount of labor in the developments. How this is effected and in what way we gain or lose in comparison to the former method is shown with some detail in Part I. It is on these lines that the developments there given are carried out.

The chief difficulties which arise in any method of treating the Lunar Theory are those connected with the determinations of the motions of the Perigee and Node. When Hill and Adams conceived the idea of the infinite determinant, and succeeded in solving one, there appeared to be an opening for the determination of these motions with an accuracy which should leave nothing to be desired. The equation for the latitude in rectangular coordinates is in a form which gives the principal part of the motion of the Node—that depending on m only—by means of an infinite determinant directly, and thence the coefficients of the periodic terms are easily obtained. But in the case of the Perigee this is not so. Several transformations are necessary before a suitable determinant is evolved, and when these have been made and the coefficients of the transformed solution obtained, the return process to find the coefficients of the periodic terms in rectangular coordinates is a laborious one. Once the motion of the Perigee has been obtained, however, we can use the equations in any form we choose.

So far the question, then, of finding the principal parts of the motion of Perigee and Node is resolved, and the coefficients dependent on these can be obtained. But we are again met by the same difficulties when we try to find further approximations to these motions. We again obtain an infinite set of linear equations. But they now contain constant terms, and though an infinite determinant can be obtained for the new part of the Perigee or Node to be found, it is in a form which does not admit of easy solution. The equations can be solved by continued approximation, but this is very laborious. The difficulty has been turned by a slight artifice which eliminates all the unknown quantities from the set of equations but that required, and the resulting equation is one which does not present any unusual difficulties in calculation.

The equations thus obtained not only showed methods by which the motions of Perigee and Node can be obtained at any stage by simple processes, but indicate the way in which Adams' theorems connecting these motions with the constant part of the expression of the Parallax arise. The remarkable proof Adams gives of his theorems is well adapted to the end in view, but it shows no indication of any possible extension to the higher terms. The methods given below, though cumbrous, have perhaps the advantage of shedding a clearer light on the theorems and of giving them a form which admits of further development. The investigation of these occupies Part II. Other theorems are also proved which may be of value when a verification of results is required.

Part III contains one or two deductions from the infinite determinant concerning the orders of the coefficients, and especially deals with the cases in which short-period terms may have large coefficients.

For convenience, the papers referred to in the following pages—

On the Lunar Theory:

I. G. W. Hill.—Researches in the Lunar Theory. American Journal of Mathematics, Vol. I, pp. 5–26, 129–147, 245–260.

II. G. W. Hill.—On the Part of the Motion of the Lunar Perigee, etc. Acta Mathematica, Vol. VIII, pp. 1–35.

III. J. C. Adams.—On the Motion of the Moon's Node in the case, etc. Monthly Notices R. A. S., Vol. XXXVIII, pp. 43–49.

IV. J. C. Adams.—Note on a Remarkable Property of the Analytical Expression, etc. Monthly Notices R. A. S., Vol. XXXVIII, pp. 460–472.

V. C. Delaunay.—Théorie de la Lune. Mémoires de l'Academie des Sciences, Vols. XXVIII, XXIX.

VI. C. Delaunay.—Note sur les mouvements du périgée et du nœud de la Lune. Comptes Rendus, Vol. LXXIV, pp. 17–21.

VII. E. W. Brown.—The Parallactic Inequalities in the Lunar Theory. American Jour. Math., Vol. XIV, pp. 141–160.

VIII. E. W. Brown.—The Elliptic Inequalities in the Lunar Theory. Amer. Jour. Math., Vol. XV, pp. 243–263, 321–336.

And on the Infinite Determinant:

IX. H. Poincaré.—Sur les déterminants d'ordre infini. Bulletin de la Société math. de France, Vol. XIV, pp. 77–90.

X. Helge von Kock.—Sur les determinants infinis, etc. Acta Math., Vol. XVI, pp. 217–295.

—are gathered together and the reference is made by means of the number in Roman numerals standing before each.

The portion of I dealing with the intermediate orbit is also found in Tisserand's Mécanique Céleste, Vol. III, Chapter XIV. The Memoir II is given in Chapter XV, and III and IV in Chapter XVI of the same volume.

PART I.

DEVELOPMENT OF THE THEORY.

1. *The Differential Equations.*

The notations to be adopted below are as follows:

n, n', the mean angular motions of the Sun and Moon about the Earth;

$2a'$, r', e', the major axis, radius vector and eccentricity of the Sun's orbit, supposed elliptic;

x, y, z, the coordinates of the Moon referred to rectangular axes, of which those of x, y are moving in the plane of the Sun's orbit with angular velocity n', and that of z is perpendicular to this plane. The positive direction of the x-axis is directed to the mean place of the Sun;

v, the solar equation of the centre;

$S = x\cos v + y\sin v$;

$u = x + y\sqrt{-1}$, $s = x - y\sqrt{-1}$, $r^2 = us = x^2 + y^2$;

$n'^2 a'^3$, the mass of the Sun;

μ, the sum of the masses of the Earth and Moon in the same units;

$\nu = n - n'$, $m = n/\nu$, $\varkappa = \mu/\nu^2$;

$$\zeta = e^{\nu(t-t_0)\sqrt{-1}}, \quad D = \zeta\frac{d}{d\zeta} = \frac{1}{\nu\sqrt{-1}}\frac{d}{dt}.$$

With the limitations here adopted—the same as those considered by Delaunay (V) and Hill (I)—the potential function is

$$\Omega = \frac{\mu}{\sqrt{r^2+z^2}} + \frac{n'^2 a'^3}{\sqrt{[r'^2 - 2Sr' + r^2 + z^2]}} - \frac{n'^2 a'^3}{r'^2} S,$$

and the equations of motion are

$$\ddot{x} - 2n'\dot{y} - n'^2 x = \frac{\partial \Omega}{\partial x},$$

$$\ddot{y} + 2n'\dot{x} - n'^2 y = \frac{\partial \Omega}{\partial y},$$

$$\ddot{z} \qquad\qquad = \frac{\partial \Omega}{\partial z}.$$

If $$\Omega' = \Omega + \tfrac{1}{2} n'^2 (x^2 + y^2) = \Omega + \tfrac{1}{2} n'^2 r^2,$$

the equations may be written

$$\left.\begin{aligned} \ddot{x} - 2n'\dot{y} &= \frac{\partial \Omega'}{\partial x}, \\ \ddot{y} + 2n'\dot{x} &= \frac{\partial \Omega'}{\partial y}, \\ \ddot{z} \qquad &= \frac{\partial \Omega'}{\partial z}. \end{aligned}\right\} \qquad (1)$$

Transforming to the independent variables u, s, z and the dependent ζ, the equations become

$$D^2 u + 2mDu = -\frac{2}{\nu^2} \cdot \frac{\partial \Omega'}{\partial s},$$

$$D^2 s - 2mDs = -\frac{2}{\nu^2} \cdot \frac{\partial \Omega'}{\partial u},$$

$$D^2 z \qquad = -\frac{1}{\nu^2} \cdot \frac{\partial \Omega'}{\partial z},$$

with the expression

$$\Omega' = \frac{\mu}{\sqrt{us + z^2}} + \frac{n'^2 a'^3}{\sqrt{r'^2 - 2Sr' + us + z^2}} - \frac{n'^2 a'^3}{r'^2} S + \tfrac{1}{2} n'^2 us.$$

This becomes by expansion

$$\frac{2}{\nu^2} \Omega' = \frac{2\mu}{\nu^2 \sqrt{us + z^2}} + m^2 us + m^2 \frac{a'^3}{r'^3} [3S^2 - us - z^2]$$
$$+ \frac{m^3}{a'} \cdot \frac{a'^4}{r'^4} [5S^3 - 3S(us + z^2)] + \ldots.$$
$$= \frac{2\kappa}{\sqrt{us + z^2}} + \tfrac{3}{4} m^2 (u + s)^2 - m^2 z^2 + \Omega_1,$$

where

$$\Omega_1 = 3m^2 \left[\frac{a'^3}{r'^3} S^2 - \tfrac{1}{4} (u + s)^2\right] - m^2 (us + z^2)\left(\frac{a'^3}{r'^3} - 1\right)$$
$$+ \frac{m^2}{a'} \cdot \frac{a'^4}{r'^4} [5S^3 - 3S(us + z^2)] + \ldots. \qquad (2)$$

It is necessary, before going further, to see in what way Ω_1 involves u, s. We have

$$\begin{aligned} S^2 &= (x \cos v + y \sin v)^2 \\ &= \tfrac{1}{4}(u+s)^2 - \tfrac{1}{2}(u^2+s^2)\sin^2 v + \frac{1}{4\sqrt{-1}}(u^2-s^2)\sin 2v, \end{aligned}$$

and if

$$\frac{a'^2}{r'^2} = 1 + \rho,$$

the first two terms of Ω_1 become

$$3m^2\left[\frac{\rho}{4}(u+s)^2 - \frac{1+\rho}{2}(u^2+s^2)\sin^2 v + \frac{1+\rho}{4\sqrt{-1}}(u^2-s^2)\sin 2v\right] - \rho m^2(us+z^2).$$

From the known properties of elliptic motion ρ, $\sin v$ are of the order e' at least. Hence Ω_1 is of the order e' or $1/a'$ at least. When we neglect $1/a'$, Ω_1 takes the form

$$\Omega_1 = \tfrac{1}{2}(Au^2 + 2Bus + Cs^2 - 2\rho m^2 z^2), \tag{3}$$

where A, B, C depend on e' and the angle $n't + \epsilon'$ only. When $1/a'$ is not neglected we can put

$$\Omega_1 = \omega_2 + \omega_3 + \ldots., \tag{3'}$$

where ω_q is a homogeneous function of u, s, z, of degree q free from fractions. Also since the time enters explicitly into Ω_1 only through the coordinates of the Sun, when $e' = 0$, Ω_1 does not contain the time explicitly.

In the general case the equations may now be written

$$\left.\begin{aligned} (D^2 + 2mD + \tfrac{3}{2}m^2)u + \tfrac{3}{2}m^2 s - \frac{\kappa u}{(us+z^2)^{\frac{3}{2}}} &= -\frac{\partial\Omega_1}{\partial s}, \\ (D^2 - 2mD + \tfrac{3}{2}m^2)s + \tfrac{3}{2}m^2 u - \frac{\kappa s}{(us+z^2)^{\frac{3}{2}}} &= -\frac{\partial\Omega_1}{\partial u}, \\ (D^2 - m^2)z \qquad\qquad\qquad - \frac{\kappa z}{(us+z^2)^{\frac{3}{2}}} &= -\tfrac{1}{2}\frac{\partial\Omega_1}{\partial z}, \end{aligned}\right\} \tag{4}$$

which are the equations to be used in developing this theory.

Taking the general value of Ω_1, multiply the first of these equations by s, the second by u and subtract, we obtain

$$D(uDs - sDu - 2mus) + \tfrac{3}{2}m^2(u^2 - s^2) = s\frac{\partial\Omega_1}{\partial s} - u\frac{\partial\Omega_1}{\partial u}. \tag{5}$$

With the same multipliers for the first two equations and the multiplier $2z$ for the last, we have by addition

$$\left.\begin{aligned} sD^2u + uD^2s + 2zD^2z + 2m\,(sDu - uDs) + \tfrac{3}{2}m^2(u+s)^2 - 2m^2z^2 & \\ - \frac{2\kappa}{(us+z^2)^{\frac{1}{2}}} = - \left(s\frac{\partial\Omega_1}{\partial s} + u\frac{\partial\Omega_1}{\partial u} + z\frac{\partial\Omega_1}{\partial z}\right) & \\ = -\sum_{q=2}^{\infty} q\omega_q \quad \text{by (3}'\text{)}. & \end{aligned}\right\} \qquad (6)$$

Again, multiplying the three equations by Ds, Du, $2Dz$ respectively and adding, we obtain

$$D\left[Du.Ds + (Dz)^2 + \tfrac{3}{4}m^2(u+s)^2 - m^2z^2 + \frac{2\kappa}{(us+z^2)^{\frac{1}{2}}}\right]$$
$$= -\left[\frac{\partial\Omega_1}{\partial s}Ds + \frac{\partial\Omega_1}{\partial u}Du + \frac{\partial\Omega_1}{\partial z}Dz\right].$$

Now in the general case Ω_1 is expressed explicitly in terms of u, s, z, t, hence

$$D\Omega_1 = \frac{\partial\Omega_1}{\partial u}Du + \frac{\partial\Omega_1}{\partial s}Ds + \frac{\partial\Omega_1}{\partial z}Dz + \frac{\partial\Omega_1}{\partial t}Dt,$$

and therefore the right-hand side of the previous equation is

$$\begin{aligned} \frac{\partial\Omega_1}{\partial t}Dt - D\Omega_1 &= \frac{1}{\nu\sqrt{-1}}\cdot\frac{\partial\Omega_1}{\partial t} - D\Omega_1 \\ &= D'\Omega_1 - D\Omega_1 \\ &= D\left[D^{-1}(D'\Omega_1) - \Omega_1\right], \end{aligned}$$

where $D'\Omega_1$ denotes the operation D performed on Ω_1 only with reference to t (or ζ) so far as it occurs explicitly in Ω_1, and D^{-1} denotes the operation inverse to D, i. e. integration with respect to ζ followed by a division by ζ.

If, then, with these substitutions we integrate, the equation becomes

$$Du.Ds + (Dz)^2 + \tfrac{3}{4}m^2(u+s)^2 - m^2z^2 + \frac{2\kappa}{(us+z^2)^{\frac{1}{2}}} = C' - \Omega_1 + D^{-1}(D'\Omega_1). \qquad (7)$$

Adding this to (6) and writing down again equations (5) and the last of equations (4), the three transformed equations of motion are

$$\left.\begin{aligned} & D^2(us+z^2) - Du.Ds - (Dz)^2 - 2m\,(uDs - sDu) + \tfrac{9}{4}m^2(u+s)^2 - 3m^2z^2 \\ & \qquad = C' - \sum_{q=2}^{\infty}(q+1)\,\omega_q + D^{-1}(D'\Omega_1), \\ & D\,(uDs - s\,Du - 2mus) + \tfrac{3}{2}m^2(u^2 - s^2) = s\frac{\partial\Omega_1}{\partial s} - u\frac{\partial\Omega_1}{\partial u}, \\ & D^2z - m^2z - \frac{\kappa z}{(us+z^2)^{\frac{3}{2}}} = -\tfrac{1}{2}\frac{\partial\Omega_1}{\partial z}. \end{aligned}\right\} \qquad (8)$$

There are some special cases to be noticed:

(i) $e' = 0$, the last term of the first equation disappears and $\omega_2 = 0$;

(ii) $\dfrac{1}{a'} = 0$, then $\omega_q = 0$, except for the value $q = 2$, $\Omega_1 = \omega_2$;

(iii) $z = 0$, the third equation disappears;

(iv) $e' = 0$, $\dfrac{1}{a'} = 0$, all the terms on the right-hand sides disappear except the constant C';

(v) $e' = 0$, $1/a' = 0$, $z = 0$, the equations reduce to those studied by Dr. Hill (I, II) and they give those inequalities which depend on e, m only.

We are going chiefly to study the equations of motion under the form (4), and it will be advisable to write down the special forms to which they reduce for zero values of certain of the constants. In the majority of the cases to be treated $1/a'$ is neglected, so that Ω_1 has the value (3) and the equations become

$$\left.\begin{aligned}(D^2 + 2mD + \tfrac{3}{2}m^2)\,u + \tfrac{3}{2}m^2 s + Bu + Cs - \frac{\varkappa u}{(us + z^2)^{\frac{3}{2}}} &= 0,\\ (D^2 - m^2)\,z - \rho m^2 z - \frac{\varkappa z}{(us + z^2)^{\frac{3}{2}}} &= 0,\end{aligned}\right\} \qquad (9)$$

(It is not necessary to write down the s-equation, since u is a complex quantity and the u-equation includes both the x-equation and the y-equation.) The particular cases of (9) are

$$z = 0, \qquad (D^2 + 2mD + \tfrac{3}{2}m^2)\,u + \tfrac{3}{2}m^2 s + Bu + Cs - \frac{\varkappa u}{(us)^{\frac{3}{2}}} = 0, \qquad 9\,\text{(i)}$$

$$z = 0,\ e' = 0, \qquad B = 0, \quad C = 0,\ \rho = 0 \text{ in equation } 9\,\text{(i)}, \qquad 9\,\text{(ii)}$$

$$e' = 0, \qquad B = 0, \quad C = 0,\ \rho = 0 \text{ in equations (9)}. \qquad 9\,\text{(iii)}$$

We shall require the equations (9) expressed in terms of x, y, z. They are

$$\left.\begin{aligned}\ddot{x} - 2n'\dot{y} - 3n'^2 x + A'x + B'y &= -\frac{\mu x}{(r^2 + z^2)^{\frac{3}{2}}},\\ \ddot{y} + 2n'\dot{x} \qquad\qquad + B'x + C'y &= -\frac{\mu y}{(r^2 + z^2)^{\frac{3}{2}}},\\ \ddot{z} + n'^2 z + n'^2 \rho z \qquad\qquad &= -\frac{\mu z}{(r^2 + z^2)^{\frac{3}{2}}},\end{aligned}\right\} \qquad (10)$$

where A', B', C', ρ are quantities depending on e' and arising from the part ω_2 of the disturbing function.

2. *Method of Development.*

We take first the equation 9 (ii). It is known (Hill, I) that in this case the equations admit of a solution which can be expressed in the form

$$\left.\begin{aligned} u &= u_0 = a_0 \Sigma a_i \zeta^{2i+1} \\ s &= s_0 = a_0 \Sigma a_{-i-1} \zeta^{2i+1} \end{aligned}\right\} \qquad i = -\infty \ldots . + \infty, \qquad (11)$$

where a_i depends on m only. This is Dr. Hill's primary solution or intermediate orbit. It contains only two arbitrary constants, while the general solution of the equation 9 (ii) contains four. It is desired to know the general solution of this equation and also of equations 9, 9 (i), 9 (iii). We shall for simplicity at present take the equation 9 (i), the extension so as to include the terms dependent on z being perfectly evident. We may write it

$$(D+m)^2 u + \tfrac{1}{2} m^2 u + \tfrac{3}{2} m^2 s - \frac{\varkappa u}{(us)^{\frac{3}{2}}} = - Bu - Cs,$$

where the terms on the right-hand side are of the order e' at the lowest. If now we put

$$u = u_0 + \delta u,$$
$$s = s_0 + \delta s\ ,$$

u_0, s_0 satisfy the equation

$$(D+m)^2 u_0 + \tfrac{1}{2} m^2 u_0 + \tfrac{3}{2} m^2 s_0 - \frac{\varkappa u_0}{(u_0 s_0)^{\frac{3}{2}}} = 0,$$

and δu, δs will therefore be determined by means of the equation

$$\begin{aligned}(D+m)^2 \delta u + \tfrac{1}{2} m^2 \delta u + \tfrac{3}{2} m^2 \delta s - \left[\frac{\varkappa}{(u_0+\delta u)^{\frac{1}{2}} (s_0+\delta s)^{\frac{3}{2}}} - \frac{\varkappa}{u_0^{\frac{1}{2}} s_0^{\frac{3}{2}}}\right] \\ = - B(u_0+\delta u) - C(s_0+\delta s). \quad (12)\end{aligned}$$

We now suppose that δu, δs are small enough for the expression in square brackets to be expanded in powers and products of these quantities.

The expansion of this portion is

$$\frac{x}{u_0^{\frac{1}{2}}s_0^{\frac{3}{2}}}\Big[-\frac{1}{2}\;\frac{\delta u}{u_0}-\frac{3}{2}\;\frac{\delta s}{s_0}+\frac{3}{8}\;\frac{(\delta u)^2}{u_0^2}+\frac{15}{8}\;\frac{(\delta s)^2}{s_0^2}$$
$$+\frac{3}{4}\;\frac{\delta u\,\delta s}{u_0 s_0}-\frac{5}{16}\;\frac{(\delta u)^3}{u_0^3}-\frac{35}{16}\;\frac{(\delta s)^3}{s_0^3}-\frac{9}{16}\;\frac{(\delta u)^2\,\delta s}{u_0^2 s_0}-\frac{15}{16}\;\frac{\delta u\,(\delta s)^2}{u_0 s_0^2}+\ldots\ldots\Big]$$

As u_0, s_0 are supposed known, we can, from their values, obtain the coefficients of the various powers of δu, δs in the form of known series. It is understood that the numerical value of m is always used, so that these series can all be expressed as odd *or* even power series in ζ with numerical coefficients. Since

$$u_0\zeta^{-1}=a_0\Sigma a_i\zeta^{2i}\;,$$
$$s_0\zeta\;=a_0\Sigma a_{-i}\zeta^{2i},$$

all the series can be expressed in terms of even positive and negative powers of ζ, so that, by a suitable arrangement, the largest coefficient is that of ζ^0. It is assumed, unless otherwise stated, that the summations include all positive and negative integral values of i, including zero.

Let

$$\tfrac{1}{2}m^2+\tfrac{1}{2}\frac{x}{(u_0s_0)^{\frac{3}{2}}}=\Sigma M_i\zeta^{2i}\;,\qquad \tfrac{3}{2}m^2\zeta^{-2}+\tfrac{3}{2}\frac{x\zeta^{-2}}{u_0^{\frac{1}{2}}s_0^{\frac{5}{2}}}=\Sigma N_i\zeta^{2i},$$
$$\frac{x\zeta}{u_0^{\frac{3}{2}}s_0^{\frac{1}{2}}}=\Sigma P_{-i}\zeta^{2i},\quad \frac{x\zeta^{-1}}{u_0^{\frac{1}{2}}s_0^{\frac{3}{2}}}=\Sigma P_i\zeta^{2i}\;,\quad \frac{x\zeta^{-3}}{u_0^{\frac{1}{2}}s_0^{\frac{7}{2}}}=\Sigma Q_i\zeta^{2i},$$
$$\frac{x\zeta^{2}}{u_0^{\frac{7}{2}}s_0^{\frac{3}{2}}}=\Sigma R_i\zeta^{2i}\;,\quad \frac{x\zeta^{-4}}{u_0^{\frac{1}{2}}s_0^{\frac{9}{2}}}=\Sigma T_i\zeta^{2i}\;,$$
$$\frac{x}{u_0^{\frac{3}{2}}s_0^{\frac{3}{2}}}=\Sigma S_i\zeta^{2i}\;,\quad \frac{x\zeta^{-2}}{u_0^{\frac{3}{2}}s_0^{\frac{7}{2}}}=\Sigma R_{-i}\zeta^{2i}.$$

When ζ^{-1} is put for ζ, u_0 changes to s_0 and s_0 to u_0. Hence $M_i=M_{-i}$ and $S_i=S_{-i}$.

As for the calculation of these, M_i is immediately obtained from the value of x/r_0^3 given by Dr. Hill (I, p. 249). For N_i we have

$$\frac{x\zeta^{-2}}{u_0^{\frac{1}{2}}s_0^{\frac{5}{2}}}=\frac{x}{r_0^3}\cdot\frac{u^2\zeta^{-2}}{r_0^2}.$$

We calculate $1/r_0^2$ and the real and imaginary parts of $u^2\zeta^{-2}$. These are easily done by the method of special values, which method will also give quickly the required coefficients. In the same way we may find the other coefficients P, Q, R.... The whole process does not entail very much labor.

The following table gives the values of M, N, P, Q:

$M_0 = +.58902\ 22856\ 4$ $\qquad M_{\pm 3} = +.00000\ 06029\ 7$

$M_{\pm 1} = +.00630\ 84231\ 2$ $\qquad M_{\pm 4} = +.00000\ 00056\ 5$

$M_{\pm 2} = +.00006\ 28883\ 4$ $\qquad M_{\pm 5} = +.00000\ 00000\ 5$

$N_0 = +1.75707\ 88032\ 7$

$N_1 = +.03686\ 55171\ 8$ $\qquad N_{-1} = +.01078\ 63527\ 2$

$N_2 = +.00054\ 79401\ 6$ $\qquad N_{-2} = +.00001\ 25690\ 4$

$N_3 = +.00000\ 70129\ 7$ $\qquad N_{-3} = +.00000\ 00982\ 3$

$N_4 = +.00000\ 00824\ 6$ $\qquad N_{-4} = +.00000\ 00007\ 6$

$N_5 = +.00000\ 00009\ 0$

$P_0 = 1.17156\ 77322$

$P_1 = +.02280\ 40093$ $\qquad P_{-1} = +.01084\ 18484$

$P_2 = +.00032\ 38766$ $\qquad P_{-2} = +.00010\ 24640$

$P_3 = +.00000\ 40164$ $\qquad P_{-3} = +.00000\ 09526$

$P_4 = +.00000\ 00465$ $\qquad P_{-4} = +.00000\ 00092$

$P_5 = +.00000\ 00005$ $\qquad P_{-5} = +.00000\ 00001$

$Q_0 = 1.17132\ 34260$

$Q_1 = +.03476\ 15314$ $\qquad Q_{-1} = -.00112\ 12092$

$Q_2 = +.00066\ 73632$ $\qquad Q_{-2} = +.00000\ 31923$

$Q_3 = +.00001\ 04714$ $\qquad Q_{-3} = +.00000\ 00337$

$Q_4 = +.00000\ 01462$ $\qquad Q_{-4} = +.00000\ 00007$

$Q_5 = +.00000\ 00020$

The numerical values of these coefficients have been given here because they are fundamental for this method of development. Further, they are the same for every series of inequalities which it may be desired to calculate. The values of M, N, P, Q given above will suffice to determine all terms in u, s, depending on e, e', $1/a'$, γ^2 as far as the first and second powers and the products, two at a time, of these quantities are concerned, and in z of the same classes of inequalities each multiplied by γ. These form by far the larger portion of the expressions given by Delaunay.

When we wish to determine the terms dependent on a' there are certain parts of the disturbing function to be added to the right-hand sides of the equations. But as these are homogeneous functions of u, s of the second, third, ... orders, their expansion is quite easy, and in order to keep the exposition as simple as possible they have not been written down.

The various cases will now be considered in the order which seems most suitable to the objects in view, namely, the development of the theory and the special theorems to be treated in Part II.

3. *The Terms whose Coefficients depend on m, e′ only.*

We take equation (12) and it is desired to find a particular solution of it such that the coefficients depend only on m, e'. As the primary solution depends only on m, and as the introduction of e' brings with it multiples of the angle $n't+\epsilon'$, no new constant of integration will be introduced. Also δu, δs will here contain e' as a factor.

At the beginning we neglect powers of e', and therefore of δu, δs above the first, and put for these terms

$$\delta u = u_\eta, \quad \delta s = s_\eta,$$

and the equation may be written

$$\zeta^{-1}(D+m)^2 u_\eta + \zeta^{-1} u_\eta . \Sigma M_i \zeta^{2i} + \zeta s_\eta . \Sigma N_i \zeta^{2i} = -B\zeta^{-1} u_0 - C\zeta^{-2}.\zeta s_0.$$

The coefficients B, C contain t only in the form $\genfrac{}{}{0pt}{}{\sin}{\cos}(n't+\epsilon')$ or in the form $\zeta^{\pm m}$. (The constant part of the angle may be omitted since we suppose that m is not commensurable with any whole number, and ϵ' plays no part in finding the coefficients of the terms in u_η, s_η.) The solution is therefore of the form

$$\begin{aligned}\zeta^{-1} u_\eta &= a_0 e' \Sigma_j \left[\eta_j \zeta^{2j+m} + \eta'_j \zeta^{2j-m}\right],\\ \zeta s_\eta &= a_0 e' \Sigma_j \left[\eta'_{-j} \zeta^{2j+m} + \eta_{-j} \zeta^{2j-m}\right],\end{aligned}$$

where j takes all positive and negative integral values including zero.

The equations which determine the coefficients η_j, η'_{-j} are

$$\begin{aligned}(2j+1+2m)^2 \eta_j + \Sigma_i M_i \eta_{j-i} + \Sigma_i N_i \eta'_{i-j} &= \text{coeff. of } \zeta^{2j+m} \text{ in } -\Sigma_j \left[Ba_j + Ca_{-j-1}\right]\zeta^{2j},\\ (2j-1)^2 \eta'_{-j} + \Sigma_i M_{-i} \eta'_{i-j} + \Sigma_i N_{-i} \eta_{j-i} &= \text{coeff. of } \zeta^{-2j-m} \text{ in } -\Sigma_j \left[Ba_j + Ca_{-j-1}\right]\zeta^{2j}.\end{aligned}$$

From these we can find η_j, η'_{-j} in terms of m by continued approximation.

The next step is to find the terms of order e'^2 by putting

$$\delta u = u_\eta + u_{\eta^2}, \quad \delta s = s_\eta + s_{\eta^2}$$

(where the terms in u, s of the order e'^2 are u_{η^2}, s_{η^2}), and separating out those terms which are of the order e'^2. The equation for them becomes, according to the previous notation,

$$\begin{aligned}\zeta^{-1}(D+m)^2 u_{\eta^2} + \zeta^{-1}u_{\eta^2}\Sigma M_i\zeta^{2i} + \zeta s_{\eta^2}\Sigma N_i\zeta^{2i} - \tfrac{3}{8}\zeta^{-2}u_\eta^2\Sigma P_{-i}\zeta^{2i} \\ - \tfrac{15}{8}\zeta^3 s_\eta^2\,\Sigma Q_i\zeta^{2i} - \tfrac{3}{4}\zeta^{-1}u_\eta . \zeta s_\eta\Sigma P_i\zeta^{2i} \\ = \text{coeff. of } e'^2 \text{ in } -\zeta^{-1}[B(u_0+u_\eta) + C(s_0+s_\eta)].\end{aligned}$$

The terms on the right-hand side and the known ones on the left-hand side necessarily contain ζ in the form

$$\Sigma\,[a_i\zeta^{2i} + \beta_i\zeta^{2i+2m} + \beta_i'\zeta^{2i-2m}],$$

the solution is therefore of the form

$$u_{\eta^2} = a_0 e'^2\Sigma_j\,[(\eta\eta')_j\zeta^{2j} + (\eta\eta)_j\zeta^{2j+2m} + (\eta'\eta')_j\zeta^{2j-2m}],$$

and a similar expression for s_{η^2}. The coefficients $(\eta\eta')_j$, etc., are the numerical quantities to be found. The method of procedure is the same as before. There are certain multiplications to be performed in the calculation of such a term as $\frac{3}{8}\zeta^{-2}u_\eta^2\Sigma P_{-i}\zeta^{2i}$, but they can be performed in two steps by first computing $\zeta u_\eta . \Sigma P_{-i}\zeta^{2i}$ (this result being wanted in the determination of the terms of order ee') and then multiplying the result by ζu_η.

We can thus proceed to find the terms dependent on all powers of e', m by successive approximation and arrive finally at a solution

$$\begin{aligned}u_{e'} &= u_0 + u_\eta + u_{\eta^2} + \ldots., \\ u_{e'} &= s_0 + s_\eta + s_{\eta^2} + \ldots.\end{aligned}$$

It may be noted that the notation used here is intended to be a suggestive one and to point at once to the order of the terms and to the power of ζ to which they belong. The letter η is associated with e' or the index m of ζ. In the coefficients, η_j is the coefficient of ζ^{2j+m} in u, $(\eta\eta)_j$ or $(\eta^2)_j$ that of ζ^{2j+2m}, etc. This method of notation for the coefficients is found to be a convenient one and it is used for the inequalities all through. For example, if $2q$ be an even positive number,

$$\begin{aligned}u_{\eta^{2q}} = a_0 e'^{2q}\Sigma_j\,[(\eta^q\eta'^q)_j\zeta^{2j} + (\eta^{q+1}\eta'^{q-1})_j\zeta^{2j+2m} + (\eta^{q-1}\eta'^{q+1})_j\zeta^{2j-2m} + \ldots. \\ + (\eta^{2q})_j\zeta^{2j+2qm} + (\eta'^{2q})_j\zeta^{2j-2qm}].\end{aligned}$$

In particular we must notice that the terms in u which do not contain the angle $n't + \epsilon'$ in their argument are

$$a_0\Sigma_j\,[a_j + e'^2(\eta\eta')_j + e'^4(\eta^2\eta'^2)_j + \ldots.]\,\zeta^{2j}. \tag{13}$$

We can suppose now, for theoretical purposes, that all the terms dependent on e', m have been calculated with the necessary accuracy, and by using the numerical value of e' in (13) we can take that for the primary value of u_0 instead of the value (11). But in the actual developments of the lunar theory this simplification has certain disadvantages. It is easier to find the extra small terms due to, for instance, the difference between (13) and (11), than to calculate the corrections to M, N which would be necessary. Hence we commence, when finding the terms depending on the lunar eccentricity or latitude or the parallax of the Sun, by using the primary orbit defined by the equations (11).

5. *The Terms whose Coefficients depend on m and the Eccentricity of the Moon's Orbit. Motion of the Perigee.*

In accordance with the remarks just made, we omit the terms depending on e', and in those results which are of a general character we shall merely state how they may be included and in what way the various expressions are altered by their presence.

We take the equations 9 (ii) and putting

$$u = u_0 + u_e + u_{e^2} + \ldots\ldots,$$
$$s = s_0 + s_e + s_{e^2} + \ldots\ldots,$$

where u_0, s_0 have the values (11), obtain for the determination of u_e, s_e

$$\zeta^{-1}(D+m)^2 u_e + \zeta^{-1} u_e . \Sigma M_i \zeta^{2i} + \zeta s_e . \Sigma N_i \zeta^{2i} = 0.$$

The solution of this equation is known to be of the form

$$\begin{aligned} \zeta^{-1} u_e &= a_0 \Sigma \left[\varepsilon_j \zeta^{2j+c} + \varepsilon'_j \zeta^{2j-c}\right], \\ \zeta s_e &= a_0 \Sigma \left[\varepsilon'_{-j} \zeta^{2j+c} + \varepsilon_{-j} \zeta^{2j-c}\right]. \end{aligned}$$

Since c is supposed incommensurable with any whole number, the constant of the argument, an arbitrary of the solution, is not put into evidence. It can be restored at any time if necessary. This remark applies generally.

When the substitution is made in the equation of motion, we obtain a set of linear equations to find ε_j, ε'_j without constant terms, and this fact necessitates a relation between them which determines the quantity c. Since the extension of Adams' theorems proved in Part II, and the results in Part III, depend on this relation to a large extent, it is necessary to develope it somewhat fully and to see in what way c is involved.

Equating to zero the coefficients of $\zeta^{2j\pm c}$, we obtain the series of linear homogeneous equations in ε_j, ε_j',

$$\left.\begin{aligned}(2j+1+m+c)^2\,\varepsilon_j &+ \Sigma_i M_i\varepsilon_{j-i} + \Sigma_i N_i\varepsilon'_{i-j} = 0,\\ (2j-1-m+c)^2\,\varepsilon'_{-j} &+ \Sigma_i M_{-i}\varepsilon'_{i-j} + \Sigma_i N_{-i}\varepsilon_{j-i} = 0.\end{aligned}\right\} \qquad (14)$$

The determinant formed by the elimination of ε_j, ε_j' we may denote by

$$\Delta(c) = \left|\begin{array}{c|c} d(c) & d \\ \hline d & d(-c) \end{array}\right|,$$

where the cross lines denote that d, $d(c)$, $d(-c)$ merely occupy the positions in the determinant assigned above. We have put $d(c)$ for

$$\begin{array}{cccccc} \cdots & \cdots & \cdots & \cdots & \cdots & \\ \vdots & (5+m+c)^2+M_0, & M_{-1}, & M_{-2}, & M_{-3}, & M_{-4}, & \vdots \\ \vdots & M_1, & (3+m+c)^2+M_0, & M_{-1}, & M_{-2}, & M_{-3}, & \vdots \\ \vdots & M_2, & M_1, & (1+m+c)^2+M_0, & M_{-1}, & M_{-2}, & \vdots \\ \vdots & M_3, & M_2, & M_1, & (-1+m+c)^2+M_0, & M_{-1}, & \vdots \\ \vdots & M_4, & M_3, & M_2, & M_1, & (-3+m+c)^2+M_0, & \vdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \end{array}$$

and $d(-c)$ for the same expression with the sign of c changed. Also d stands for

$$\begin{array}{ccccccc} \cdots & \cdots & \cdots & \cdots & \cdots & \\ \vdots & N_0, & N_{-1}, & N_{-2}, & N_{-3}, & N_{-4}, & \vdots \\ \vdots & N_1, & N_0, & N_{-1}, & N_{-2}, & N_{-3}, & \vdots \\ \vdots & N_2, & N_1, & N_0, & N_{-1}, & N_{-2}, & \vdots \\ \vdots & N_3, & N_2, & N_1, & N_0, & N_{-1}, & \vdots \\ \vdots & N_4, & N_3, & N_2, & N_1, & N_0, & \vdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \end{array}$$

When these are placed in the expression for $\Delta(c)$ in the positions assigned and the cross lines taken away, we have the required determinant.

This is a third form of the infinite determinant giving c. It differs from that obtained by Dr. Hill (II) in being what may be called "doubly-infinite," that is, it is infinite towards the directions not only of the outer sides of the square but also towards those of the cross lines. In this feature it resembles that which I obtained (VIII) by the use of the equations (8), but differs in one most important aspect, namely, that by a suitable system of divisors it can be put into a convergent form.

I repeat the observations there made as to the number of roots and the relations which hold between them. Denoting by q the number of coefficients of the series $s_1, s_3, \ldots . s_\infty$, there are $4q + 2$ unknowns in the $4q + 2$ equations; the determinant has $4q + 2$ rows and columns, and it forms an equation for c of the order $2(4q + 2)$. If c_1, c_3 be two roots of this equation not differing by an even integer, the whole series of roots is

$$\pm c_1, \ \pm(c_1 \pm 2), \ \pm(c_1 \pm 4), \ldots,$$
$$\pm c_3, \ \pm(c_3 \pm 2), \ \pm(c_3 \pm 4), \ldots.$$

Either c_1 or c_3 is shown to be zero, since

$$u_e = Du, \quad s_e = Ds$$

is a solution of the equations for u_e, s_e.

Now these two sets of roots are also roots of the equation

$$\left(\sin^2\frac{\pi c}{2} - \sin^2\frac{\pi c_1}{2}\right)\left(\sin^2\frac{\pi c}{2} - \sin^2\frac{\pi c_3}{2}\right) = 0,$$

or of the equation

$$(\cos \pi c - \cos \pi c_1)(\cos \pi c - 1) = 0.$$

Hence we have

$$\Delta(c) \equiv A(\cos \pi c - \cos \pi c_1)(\cos \pi c - 1),$$

an identical equation in which A is independent of c. We most easily find the constant A by equating the coefficients of the highest power of c in each member of the equation. In $\Delta(c)$ this is unity, and in the right-hand side it is

$$A\,\frac{4^2}{1^2}\cdot\frac{4^2}{9^2}\cdot\frac{4^2}{25^2}\cdots\cdot = A\left(\frac{4}{1}\cdot\frac{4}{3.5}\cdot\frac{4}{7.9}\cdots\cdot\right)^4$$
$$= A\left(\frac{4}{1}\cdot\frac{4}{4^2-1}\cdot\frac{4}{8^2-1}\cdots\cdot\right)^4,$$

and therefore

$$(\cos \pi c - \cos \pi c_1)(\cos \pi c - 1) \equiv \Delta(c) \times \left(\frac{4}{1}\cdot\frac{4}{4^2-1}\cdot\frac{4}{8^2-1}\cdots\right)^4$$
$$\equiv \nabla(c).$$

The new determinant $\nabla(c)$ is put into a convenient form by multiplying the middle rows of the upper half of $\Delta(c)$ by 4, the two rows on either side of this by $\frac{4}{4^4-1}$, the next two rows above and below by $\frac{4}{8^2-1}$ and so on. The lower half being treated in the same way, all the divisors are accounted for. We might then proceed as Dr. Hill has done to obtain the determinant in a better form, with every constituent in the central diagonal unity. The new determinant and

$\nabla(c)$ are both convergent, that is, in each case the product of the constituents of the central diagonal is finite and the sum of the other constituents is finite. (Poincaré IX, Helge von Koch X). But since the determinant will not be used for finding c, and since it has been treated as far as is necessary for the purposes of this paper, it does not seem to be of advantage to develope it further. Two points are of importance in what follows: firstly, *the determinant* $\Delta(c)$ *is a symmetrical one*, and secondly, in consequence of this and also since it vanishes for the special value of c which denotes the principal part of the motion of the lunar Perigee, *the coefficients* ε_j, ε'_j *are proportional to the first minors of the constituents of any row or column.*

It is not difficult to see what changes are made by the introduction of the terms dependent on e'^2 and its powers. Instead of using the values (11) of u_0, s_0, we must use the values (13); also, instead of M_i, N_i we shall have

$$(M)_i = M_i + \text{terms containing } e'^2 \text{ and its powers},$$
$$(N)_i = N_i + \text{`` \quad `` \quad `` \quad `` \quad ``}$$

No other alteration is necessary in the equations since the terms arising from A, B, C [equation (3)] can be included in (M), (N). We, however, do not have $(M)_i = (M)_{-i}$. All the results concerning the infinite determinant for c still hold. Let for brevity

$$(\varepsilon)_j = \varepsilon_j\,(1 + \text{terms containing } e'^2 \text{ and its powers}),$$
$$(\varepsilon')_j = \varepsilon'_j\,(1 + \text{`` \quad `` \quad `` \quad `` \quad ``}),$$
$$(c) = c + \text{terms containing } e'^2 \text{ and its powers}$$

be the new values. Then as before $(\varepsilon)_j$, $(\varepsilon')_j$ are proportional to the minors of the constituents of any row *or* column of the new determinant, which remains symmetrical.

It remains to state that when c has been found, all the coefficients ε_i, ε'_i can be found in terms of one of them by any suitable method of continued approximation. The minors of the determinant do not seem adapted to calculation. It is found that $\varepsilon_0 - \varepsilon'_0$ is approximately $2e$ where e is Delaunay's eccentricity, and therefore this quantity is a suitable arbitrary. The letter ε is used in all coefficients involving the lunar eccentricity just as the letter η is in all coefficients involving the solar eccentricity.

5. *Further Approximations to the Terms dependent on the Eccentricity of the Moon's orbit and to the Motion of the Perigee.*

The next process is to find the terms of the order e^2. These follow in the same way from the results just obtained as those of order e'^2 did from those of order e'. If the part of c depending on e, before neglected, be δc, we can easily see that δc is of the order e^2 and that we need not know it to obtain the coefficients of the order e^2. We suppose the coefficients of the order e^2 found.

We proceed, then, to find the terms dependent on e^3 with the corresponding new part of c. We are only going to consider the terms depending on the indices $2j \pm (c + \delta c)$ of ζ, since it is by these terms alone that δc is determined. Stopping at the third order we put

$$u = u_0 + u_e + u_{e^2} + u_{e^3},$$

where

$$u_e + u_{e^3} = a_0 \Sigma_j \left[\varepsilon_j \zeta^{2j+c+\delta c} + \varepsilon'_j \zeta^{2j-c-\delta c} + (\varepsilon^2 \varepsilon')_j \zeta^{2j+c+\delta c} + (\varepsilon \varepsilon'^2)_j \zeta^{2j-c-\delta c} \right],$$

and similar expressions for the complementary variable. The last two coefficients are of the order e^3, and therefore on substitution in the equation of motion we can put c for $c + \delta c$ when this quantity appears as a factor of these coefficients. We obtain, by separating out the terms of the order e^3,

$$\begin{aligned} &[(2j + 1 + m + c + \delta c)^2 - (2j + 1 + m + c)^2]\, \varepsilon_j \\ &\quad + (2j + 1 + m + c)^2 (\varepsilon^2 \varepsilon')_j + \Sigma_i M_i (\varepsilon^2 \varepsilon')_{j-i} + \Sigma_i N_i (\varepsilon \varepsilon'^2)_{i-j} \\ &\quad = \text{coeff. of } \zeta^{2j+c+\delta c} \text{ in } \frac{\varkappa \zeta^{-1}}{a_0} (u_0 + u_e + u_{e^2})^{-\frac{1}{2}} (s_0 + s_e + s_{e^2})^{-\frac{3}{2}} \text{ of the order } e^3, \end{aligned}$$

and a similar equation. Omitting negligible quantities on the left-hand side, the equation becomes

$$\begin{aligned} &2\delta c\, (2j + 1 + m + c)\, \varepsilon_j + (2j + 1 + m + c)^2 (\varepsilon^2 \varepsilon')_j + \Sigma_i M_i (\varepsilon^2 \varepsilon')_{j-i} + \Sigma_i N_i (\varepsilon \varepsilon'^2)_{i-j} \\ &\qquad = \text{same expression as before.} \qquad (15) \end{aligned}$$

Similarly the other equation gives

$$\begin{aligned} &2\delta c\, (2j - 1 - m + c)\, \varepsilon'_{-j} + (2j - 1 - m + c)(\varepsilon \varepsilon'^2)_{-j} + \Sigma_i M_{-i} (\varepsilon \varepsilon'^2)_{i-j} + \Sigma_i N_{-i} (\varepsilon^2 \varepsilon')_{j-i} \\ &\quad = \text{coeff. of } \zeta^{-2j-c-\delta c} \text{ in } \frac{\varkappa \zeta^{-1}}{a_0} (u_0 + u_e + u_{e^2})^{-\frac{1}{2}} (s_0 + s_e + s_{e^2})^{-\frac{3}{2}} \text{ of the order } e^3. \qquad (15) \end{aligned}$$

The terms on the right-hand sides of these equations are known, those on the left-hand sides are to be determined. We can get an equation of the first order for δc free from the as yet undetermined coefficients $(\varepsilon^2\varepsilon')$, $(\varepsilon\varepsilon'^2)$. There are, in the notation of the last section, $4q+2$ equations and $4q+3$ unknowns to be determined. One of these is arbitrary, and it involves a new definition of the "eccentricity constant" of integration. We may define this as may be most convenient. At present we proceed to the determination of δc, which is not affected by this observation.

Multiply the first of equations (15) by ε_j, the second by ε'_{-j}, and sum the two for all values of j. Now, in these equations the coefficients of $(\varepsilon^2\varepsilon')_j$, $(\varepsilon\varepsilon'^2)_{-j}$ are exactly the same, term for term, as those of ε_j, ε'_{-j} in equations (14). If, therefore, we multiply the equations (15) by the respective minors of $\Delta(c)$, all terms involving $(\varepsilon^2\varepsilon')_j$, $(\varepsilon\varepsilon'^2)_{-j}$ will disappear. But as these minors are proportional to ε_j, ε'_{-j}, the same object will be attained if we multiply by ε_j, ε'_{-j}. If, however, we conceive of difficulties arising from the fact that $\Delta(c)$ is not in a convergent form and therefore that its minors are infinite quantities, though their ratios are finite, we can proceed as follows. After the multiplication by ε_j, ε'_{-j} and the addition of the equations for all values of j, the coefficient of $(\varepsilon^2\varepsilon')_i$ is

$$(2i+1+m+c)^2\,\varepsilon_i + \Sigma_j M_j\varepsilon_{i-j} + \Sigma_j N_j\varepsilon'_{j-i},$$

which, by equations (14), is zero. And so for all the other similar terms. This process is, of course, algebraically identical with the previous one.

Hence we have

$$\begin{aligned}2\delta c\Sigma_j\left[(2j+1+m+c)\,\varepsilon_j^2 + (2j-1-m+c)\,\varepsilon'^2_{-j}\right]&\\ =\Sigma_j\Big[\varepsilon_j\times\text{ coeff. of }\zeta^{2j+c+\delta c}\text{ in }\frac{\varkappa\zeta^{-1}}{a_0}\,&(u_0+u_e+u_{e^2})^{-\frac{1}{2}}(s_0+s_e+s_{e^2})^{-\frac{3}{2}}\\ +\,\varepsilon'_{-j}\times\text{ coeff. of }\zeta^{-2j-c-\delta c}\text{ in }\frac{\varkappa\zeta^{-1}}{a_0}\,&(u_0+u_e+u_{e^2})^{-\frac{1}{2}}(s_0+s_e+s_{e^2})\Big]^{-\frac{3}{2}},\end{aligned}$$

where the terms on the right-hand side of the order e^4 alone are considered. But

$$s_e\zeta = a_0\Sigma_j\left[\varepsilon_j\zeta^{-2j-c-\delta c} + \varepsilon'_{-j}\zeta^{2j+c+\delta c}\right],$$

and therefore, using this on the right-hand side of the previous equation, we obtain

$$\begin{aligned}2\delta c\Sigma_j\left[(2j+1+m+c)\,\varepsilon_j^2 + (2j-1-m+c)\,\varepsilon'^2_{-j}\right]&\\ =\text{const. part, order }e^4\text{, in }\frac{\varkappa}{a_0^2}\cdot&\frac{(u_0+u_e+u_{e^2})\,s_e}{(u_0+u_e+u_{e^2})^{\frac{3}{2}}(s_0+s_e+s_{e^2})^{\frac{3}{2}}}.\qquad(16)\end{aligned}$$

Transforming to real coordinates and putting

$$X_{e^2} = x_0 + x_e + x_{e^2}, \quad Y_{e^2} = y_0 + y_e + y_{e^2}, \quad R_{e^2}^2 = X_{e^2}^2 + Y_{e^2}^2,$$

the right-hand side becomes

$$\text{const. part, order } e^4 \text{, in expansion of } \frac{\kappa}{a_0^2} \cdot \frac{X_{e^2} x_e + Y_{e^2} y_e}{R_{e^2}^3}, \tag{16'}$$

an expression of somewhat remarkable form and simplicity. When first obtained, it seemed to suggest that Adams' theorems (IV) as to the connection between the constant parts of the Parallax of the Moon and certain parts of the motions of the Perigee and Node, must really arise naturally from this mode of development of the lunar theory. After some trouble I succeeded in proving again these theorems and extending them so as to show them not merely in the form of ratios but in an exact form. They will be given in Part II of this paper.

It is easy to extend these results so as to include all powers of e'. When we do so the formula (16) or (16′) becomes

$$2\delta(c)\Sigma_j\left[\{2j + 1 + m + (c)\}(\varepsilon)_j^2 + (2j - 1 - m + c)(\varepsilon')_{-j}^2\right]$$
$$= \text{const. part, order } e^4 \text{, in } \frac{\kappa}{a_0^2} \cdot \frac{(X)_{e^2}(x)_e + (Y)_{e^2}(x)_e}{(R)_{e^2}^3}, \tag{17}$$

where the brackets round the various symbols have the same meaning as that given at the end of section 4.

The coefficient of δc or $\delta(c)$ remains the same if we are finding the terms in the motion of Perigee dependent on the latitude of the Moon or the parallax of the Sun. If we are in the process of finding $\delta(c)$, that part that depends on e'^2 can be obtained as in equation (16) by finding the constant of order $e^2 e'^2$ in the expansion of

$$\frac{\kappa}{a_0^2} \cdot \frac{(X)_e x_e + (Y)_e y_e}{(R)_{e^2}^3},$$

where $(X)_e$, $(Y)_e$, $(R)_e$ contain merely the terms, supposed to be previously found, dependent on e, e', e'^2, ee' and m. There are also the terms arising from ω_2 to be included. So that in all cases the determination of the new part of c depends only on an equation which contains known terms found in previous approximations.

Similar results will follow for the terms dependent on higher powers of e^2, e'^2 in c. For example, the equation necessary to find the part in c of the order e^4 may be easily obtained by looking at equations (15). It is $\delta^2 c$ where $\delta^2 c$ is given by

$$\left.\begin{aligned} &2\delta^2 c\,\Sigma_j\left[(2j+1+m+c)\,\varepsilon_j^2+(2j-1-m+c)\,\varepsilon'^2_{-j}\right]+(\delta c)^2\,\Sigma_j\left[\varepsilon_j^2+\varepsilon'^2_{-j}\right]\\ &\quad+\Sigma_j\left[(2j+1+m+c)^2\,\varepsilon_j(\varepsilon^2\varepsilon')_j+(2j-1-m+c)^2\,\varepsilon'_{-j}(\varepsilon\varepsilon'^2)_{-j}\right]\\ &\qquad=\text{const. coeff., order } e^4, \text{ in } \frac{\kappa}{a_0^2}\cdot\frac{X_{e^4}x_e+Y_{e^4}y_e}{R_{e^4}^3}, \end{aligned}\right\} \quad (18)$$

where $$X_{e^4}=x_0+x_e+x_{e^2}+x_{e^3}+x_{e^4}, \text{ etc.}$$

The coefficients to be determined therewith by the equations from which this expression arises, are those of the order e^5 with arguments $(2iD \pm l)$, where $2D$ is the argument of the Variation and l that of the principal elliptic term.

If we wish to determine the terms dependent on the square of the parallax of the Sun, similar results will follow. On the right-hand sides of the equations we shall have further terms arising from Ω_1, but these terms will be all known by previous computations, so that the general result for all terms in the motion of the perigee is this: *In any stage of the approximations we can always find the new part of the motion of the perigee by simple computations without being compelled either to solve an infinite determinant or to find by successive approximation the new coefficients arising therewith. The new coefficients can then be found by the usual methods.*

These results hold for any terms whatever in the motion of the perigee, and also, as will be seen immediately, for those in the motion of the node. The most important feature in them all is that just stated in italics. The solution of an infinite determinant, in whatever way performed, is very laborious, and the continued approximation method applied to δc and to the coefficients simultaneously, is not much less so. The method here given, when once the parts dependent on m only have been calculated, avoid these laborious processes—an important point quite independent of its mathematical interest. It is not to be forgotten also that in one respect theory here goes hand in hand with observation. The motions of perigee and node are now capable of being determined by observation with much greater accuracy than the majority of the coefficients of the periodic terms. In the method outlined above theory also gives the new parts of the motions before the coefficients of the corresponding periodic terms.

We now proceed to find the effect of including in our equations the motion of the Moon in latitude,

6. *Terms dependent on the Latitude of the Moon. Motion of the Node.*

For these terms we use equations 9 (iii), omitting those dependent on the solar eccentricity. Since z is a small quantity in powers of which we suppose that expressions are possible, and since the second of these equations contains it in its first, third, fifth, ... powers only, while the first equation contains it in its second, fourth, ... powers only, we are able to treat the equations separately, going first to one and then to the other as we obtain each new approximation. This fact facilitates the performance of the actual calculations very largely.

Let γ denote generally a quantity of the same order as z. The z-equation is

$$D^2z-\left(m^2+\frac{\varkappa}{(r^2+z^2)^{\frac{3}{2}}}\right)z=0.$$

Let

$$\left.\begin{aligned} z&=z_\gamma+z_{\gamma^3}+z_{\gamma^5}+\ldots\ldots,\\ u&=u_0+u_{\gamma^2}+u_{\gamma^4}+\ldots\ldots,\\ s&=s_0+s_{\gamma^2}+s_{\gamma^4}+\ldots\ldots\end{aligned}\right\}\tag{19}$$

Expanding the last term in the z-equation, we have

$$D^2z-\left(m^2+\frac{\varkappa}{r^3}\right)z=-\frac{3}{2}\,\frac{\varkappa z^3}{r^5}+\frac{15}{8}\,\frac{\varkappa z^5}{r^7}+\ldots\ldots$$

The equation for z_γ will be

$$D^2z_\gamma-\left(m^2+\frac{\varkappa}{r_0^3}\right)z_\gamma=0.\tag{20}$$

The solution of this equation can be put into the form

$$z_\gamma\sqrt{-1}=a_0\Sigma\left[K_j\zeta^{2j+g}+K'_{-j}\zeta^{-2j-g}\right],\qquad j=+\infty\ldots\ldots-\infty,\tag{21}$$

where g must be found, and K_j, K'_{-j} are of the order γ and such that $K_j=-K'_{-j}$. The arbitrary constant attached to the new angle which the presence of g introduces can be understood, as in section 4, to be present.

It is known from Adams' work (III) that g is found from an infinite determinant. Expanding $\varkappa/r_0^3$, we have from section 2

$$D^2z_\gamma-2z_\gamma\Sigma M_i\zeta^{2i}=0,$$

and therefore on substitution of the assumed value of z_γ and on equating the coefficients of ζ^{2j+g} to zero,

$$(2j+g)^2K_j-2\Sigma K_iM_{j-i}=0.\tag{22}$$

This set of equations gives a symmetrical infinite determinant which, after certain transformations, was calculated by Adams (III) to find g. The coefficients K_j are proportional to the first minors of the constituents of any row *or* column of this determinant. When g is known and we have thence obtained the ratios of K_j to any one of them, say K_0, which we take as the arbitrary constant of the solution, we substitute the resulting values of z in the first of equations 9 (iii) and find by the same methods as before the values of u_{γ^2}, s_{γ^2}, which, as the suffixes indicate, are of the order γ^2.

Returning with these to the z-equation, it is desired to find the terms in z of the order γ^3 together with those in g of the order γ^2. Confining our attention to the latter object, we assume

$$(z_\gamma + z_{\gamma^3})\sqrt{-1} = a_0 \Sigma\left[\{K_j + (K^2K')_j\}\,\zeta^{2j+g+\delta g} + \{K'_{-j} + (KK'^2)_{-j}\}\,\zeta^{-2j-g-\delta g}\right].$$

We omit the terms which depend on the indices $2i \pm 3g$, as their coefficients can be determined to the order γ^3 with the known value of g. We have $(K^2K')_j = -(KK'^2)_{-j}$. Equating to zero the coefficients of $\zeta^{2j+g+\delta g}$ for all values of j, we get a set of equations for the determination of the new coefficients and δg. Let

$$X_{\gamma^2} = x_0 + x_{\gamma^2}, \quad Y_{\gamma^2} = y_0 + y_{\gamma^2}, \quad R^2_{\gamma^2} = X^2_{\gamma^2} + Y^2_{\gamma^2}, \tag{23}$$

then, by reasoning similar to that used in the determination of δc, the set of equations may be written

$$2\delta g\,(2j + g)\,K_j + (2j + g)^2 (K^2K')_j - 2\Sigma M_{j-i}\,(K^2K')_i$$
$$= \text{coeff., order } \gamma^3, \text{ of } \zeta^{2j+g+\delta g} \text{ in } \frac{\kappa z_\gamma \sqrt{-1}}{a_0\,(R^2_{\gamma^2} + z^2_\gamma)^{\frac{3}{2}}}. \tag{24}$$

Multiplying this equation by K_j, and summing the set of equations so treated for all values of j, we have by equation (22) (as in the last section) the new coefficients disappearing and

$$2\delta g\,\Sigma\,(2j + g)\,K_j^2 = \left[\Sigma\,K_j \times \text{coeff., order } \gamma^4, \text{ of } \zeta^{2j+g+\delta g} \text{ in } \frac{\kappa z_\gamma \sqrt{-1}}{a_0\,(R^2_{\gamma^2} + z^2_\gamma)^{\frac{3}{2}}}\right]. \tag{25}$$

Similarly,

$$2\delta g\,\Sigma\,(2j + g)\,K'^2_{-j} = \Sigma\left[K'_{-j} \times \text{coeff., order } \gamma^4, \text{ of } \zeta^{-2j-g-\delta g} \text{ in } \frac{\kappa z_\gamma \sqrt{-1}}{a_0\,(R^2_{\gamma^2} + z^2_\gamma)^{\frac{3}{2}}}\right]. \tag{25}$$

Now, since $K'_{-j} = -K_j$, and therefore

$$-z_\gamma\sqrt{-1} = a_0 \Sigma\left[K_j \zeta^{-2j-g-\delta g} + K'_{-j}\zeta^{2j+g+\delta g}\right],$$

we obtain, by addition of the two previous equations,

$$4\delta g\,\Sigma\,(2j+g)\,K_j^2 = \text{const. coeff., order } \gamma^4, \text{ in } \frac{\kappa z_\gamma^2}{a_0^2\,(R_{\gamma^2}^2+z_\gamma^2)^{\frac{3}{2}}}, \tag{26}$$

which is the formula for obtaining δg.

This formula is easily extended so as to include all powers of e'. In determining K_j, instead of using r_0 we use (r_0), where the notation is that used in section 5. Let the corresponding values of K_j, g, δg, etc., be $(K)_j$, (g), $\delta(g)$, etc. The equation then becomes

$$4\delta(g)\,\Sigma\,\{2j+(g)\}\,(K)_j^2 = \text{const. coeff., order } \gamma^4, \text{ in } \frac{\kappa\,(z)_\gamma^2}{a_0^2\,\{(R)_{\gamma^2}^2+(z)_\gamma^2\}^{\frac{3}{2}}}. \tag{27}$$

The part of the motion of the node which is of the order e^2, and that of the motion of the perigee which is of the order γ^2, will be required for the investigations in Part II. They are not difficult to write down if we look at the results (16′) and (26). Let $z_{e\gamma}$ be the part of z of the order $e\gamma$. This can be found when we know the principal parts of the motions of perigee and node only. It can then be easily seen that the part of δg depending on e^2 is given by means of the equation

$$4\delta g\,\Sigma_j\,(2j+g)\,K_j^2 = \text{const coeff., order } e^2\gamma^2, \text{ in } \frac{\kappa z_\gamma (z_\gamma + z_{e\gamma})}{a_0^2\,R_{e^2}^2}. \tag{28}$$

Also that the part of δc of the order γ^2 is given by the equation

$$\begin{aligned}&2\delta c\,\Sigma_j\left[(2j+1+m+c)\,\varepsilon_j^2+(2j-1-m+c)\,\varepsilon'^2_{-j}\right]\\ &\quad = \text{const. part., order } e^2\gamma^2, \text{ in } \frac{\kappa}{a_0^2}\cdot\frac{(X_e+x_{\gamma^2})\,x_e+(Y_e+y_{\gamma^2})\,y_e}{[(X_e+x_{\gamma^2})^2+(Y_e+y_{\gamma^2})^2+(z_\gamma+z_{e\gamma})^2]^{\frac{3}{2}}}.\end{aligned} \tag{29}$$

Finally, by putting brackets ( ) round each letter, we may include as before in the formulæ (28) and (29), all terms dependent on e'^2 and its powers.

The equations (28) and (29), with powers of e'^2 included, will be named (30) and (31) respectively. (30), (31)

It is unnecessary to go further with these expansions. Sufficient has been said to indicate the manner of treating the whole Lunar Theory after this method. The terms which depend on the parallax of the Sun are a little troublesome owing to the new terms which arise from Ω_1, but they are much less so than when treated by the simultaneous equations (8), since, for a given degree of approximation, they are one order lower in u, s in the former case than in the latter. Nevertheless, should it be considered that in finding the terms of high orders in e, e', γ, $1/a'$ the method indicated above becomes too troublesome, nothing prevents us from returning to equations (8), these being available at any stage. No increase of labor results from the mere change of method.

After the experience gained in the calculations of the coefficients made up to the present time by both methods, it seems probable that the equations (8) will involve less labor than (9) for terms of the fourth order in e, e', γ^2, $1/a'$, if not for the third. Nothing even then stands in the way of the determination of the new parts of c and g by the methods just exhibited.

7. *Order of Proceeding.*

Having resolved the chief difficulties in the theory, namely, the determination of the motions of Perigee and Node, it is possible to give a consistent plan for the computation of the complete expressions of the coordinates. The order which seems to arise most naturally from the previous results may be sketched out as follows:

Terms dependent on

(i) m only.
(ii) m and e', e'^2, e'^3 .... in succession.
(iii) m and e, ee', ee'^2, ... , e^2, e^2e', $e^2e'^2$, ... , e^3, e^3e', ...
(iv) m and γ, $\gamma e'$, $\gamma e'^2$, ... , γ^2, γ^2e', ... , γ^3, ...
(v) m and any other combinations of γ, e, e'.
(vi) m and $1/a'$ and combinations with γ, e, e'.

This order may be varied somewhat.

The arguments in Delaunay's notation are, $2D$ that of the variation, l that of the principal elliptic inequality, l' that of the annual equation, F that of the principal term in latitude. The expressions of the coordinates should then be arranged in the order

$$2iD,\ 2iD \pm l',\ 2iD \pm 2l',\ \ldots.,\quad 2iD \pm l,\ 2iD \pm l \pm l',\ 2iD \pm l \pm 2l',\ \ldots.,$$

and so on. The chief difference from Delaunay's arrangement being that the terms whose arguments differ only by multiples of $2D$ are to be placed together in all cases.

It is proposed to carry the computations sufficiently far in x, y, z to enable the coefficients in longitude and latitude to be found accurately to one thousandth of a second of arc. This makes it necessary to carry some of the earlier calculations to a somewhat greater degree of accuracy than is actually needed for the special coefficients. But in any case the terms in m only, and the principal parts of the motions of perigee and node, carried to 15 places of decimals by Hill and Adams (I, II, III), are amply sufficient.

8. *The Arbitrary Constants.*

The constants which arise in the arguments by the integrations are the same as those of Delaunay. Those outside of the signs, sine and cosine have been referred to generally as a_0, e, γ. As for a_0, we can define it in such a way that the determination of the coefficients is made as simple as possible, and this will cause in general a determination of a new part to a_0 whenever we are making an approximation which contains a constant term. It would seem better, however, to define a_0 as that value given by Dr. Hill (I) in terms of $(\mu/n^2)^{\frac{1}{3}}$ and m, making the necessary changes after each approximation when the latter has been made under the previous method. Similarly, it seems the best to define the "eccentricity" constant so that it is the coefficient of $\sin l$ in x/a_0, and the latitude constant so that it is the coefficient of $\sin F$ in z/a_0. Then a_0 differs from Delaunay's a by terms which vanish with m; the first approximation (neglecting m and all powers and products of the constants) to the eccentricity constant is $2e$, where e is Delaunay's constant; and the similar first approximation to the latitude constant is the quantity defined by Delaunay as γ.

The numerical value of m is used all through, the other constants being left arbitrary.

9. *Verification of the Results.*

For verification purposes we have several equations at our disposal. The most useful of these when the coefficients of ζ^{2j} are not under consideration, is the second of equations (8). If it be desired to verify the coefficients of $\zeta^{2j+\Lambda}$, where Λ is some linear combination of m, c, g, we substitute in this equation the results obtained and find therefrom an equation of the form

$$\lambda\zeta^{\Lambda} + \lambda'\zeta^{-\Lambda} = 0,$$

where λ, λ' contain integral powers of ζ only. The coefficients of the various powers of ζ ought to vanish identically after substitution, and therefore if we put $\zeta = 1$ in λ, λ', the results ought equally to be zero identically. When $\zeta = 1$, $\lambda + \lambda' = 0$, and therefore this method will not be available when Λ is zero. In this equation of verification all the terms are of the second degree in u, s, and the combinations of the coefficients are totally different from those arising from equations (4) used for finding them.

As for the coefficients of ζ^{2j}, the results proved in Part II concerning the constant part of $1/r$ used in connection with equation (6) will verify all coefficients of the orders e^2, γ^2, e^4, $e^2\gamma^2$. γ^4 and all powers e'^2. For practical purposes, then, we have only the terms of orders e'^2, e'^4, ... $1/a'^2$, e^2/a'^2, ..., etc., outstanding. Equation (6) can be used for these by expanding $2x(us + z^2)$, and, after choosing out the terms of the required order, putting $\zeta = 1$.

10. *General Observations.*

It will be well to contrast the method here outlined with that used in the development of the Elliptic and Parallactic Inequalities (VII, VIII). It must be premised that both methods are built up on Dr. Hill's primary solution. These remarks refer merely to the best method of continuation.

The chief disadvantage of the present treatment lies in the frequent multiplication of series of ζ with numerical coefficients. Although every multiplication is available for several classes of inequalities, besides the one for which it was especially made, in terms of high order these computations will become very tedious. But there are several advantages which may render the method worthy of consideration. In the first place there are no numerical coefficients to be computed after the multiplication of series has been performed, like those arising from the use of equations (8). In the latter, for instance, when finding the terms of order e^2 we have to compute

$$\Sigma_j (j,\ i,\ 0,\ 1)\, \varepsilon_j \varepsilon'_{j-i}$$

for all values of i, the portion in brackets being a somewhat complicated function of m and c, and different for different values of i, j (VIII, p. 322). Such calculations are fruitful in producing errors. Secondly, the terms arising from Ω_1 are very simple and, when $1/a'$ is neglected, of the first order in u, s, so that only a very few short operations are necessary. Thirdly—perhaps the most important point to be mentioned—the multiplications of series, being always of the same nature, can be performed by a computer from one general formula. This would require little extra knowledge on his part beyond that of logarithms, and his results can be checked by a computation with a special value of ζ.

The method here outlined has been successfully employed in computing the coefficients of orders e', e'^2, ee' with the numerical value of m;* much labor apparently was saved by the readiness with which the work was reduced to easily remembered operations and by the diminished opportunities of making errors in calculation. Certain difficulties owing to the near equality of the coefficients of $\zeta^{\pm m}$ and $\zeta^{\pm(2-m)}$ and also to the large divisors, were more readily bridged. I hope shortly to test the terms in e^2, and also the part of the motion of the perigee of the order e^2 (VIII, p. 337). This last differed somewhat from Delaunay's value (VI), and before it can be accepted, ought to have a further test applied and the errors, if any, corrected.

PART II.

THEOREMS CONCERNING THE CONSTANT PART IN THE EXPRESSION OF THE PARALLAX OF THE MOON.

In the investigations which follow, the Parallax of the Sun is neglected. With this exception the results obtained are perfectly general.

Let
$$x = X_q, \quad y = Y_q, \quad z = Z_q$$
be the general solution of equations (9) or (10), neglecting quantities in the coefficients beyond the order $e^p\gamma^{q-p}$ $(p = 0, 1, 2 \ldots . q)$. We define e, γ as the arbitrary constants of integration in eccentricity and latitude, of the same order of small quantities as those used by Delaunay.

Let
$$\left.\begin{aligned} X_q &= x_0 + x_1 + \ldots . + x_q, \\ Y_q &= y_0 + y_1 + \ldots . + y_q, \\ Z_q &= z_1 + z_2 + \ldots . + z_q, \\ R_q^2 &= X_q^2 + Y_q^2 + Z_{q-1}^2 \quad , \end{aligned}\right\} \tag{32}$$
where x_q, y_q are of the form $\alpha_q e^q + \alpha_{q-2} e^{q-2}\gamma^2 + \alpha_{q-4} e^{q-4}\gamma^4 + \ldots .$,
and z_q is of the form $\alpha_{q-1} e^{q-1}\gamma + \alpha_{q-3} e^{q-3}\gamma^3 + \ldots .$

From what we know of the expressions of the coordinates of the Moon, α_q, α_{q-2}, ... are functions of m, e' and of the angles $2\nu(t - t_0)$, $n't + \epsilon'$,

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* Monthly Not. R. A. S., Vols. LIV, p. 471; LV, pp. 3–5.

$c\nu(t-t_1)$, $2g\nu(t-t_2)$ and $\alpha_{q-1}, \alpha_{q-3} \ldots.$ of m, e' and the angles $2\nu(t-t_0)$, $n't+\epsilon'$, $c\nu(t-t_1)$ and odd multiples of $g\nu(t-t_2)$. According to the notation previously used there should be brackets ( ) round all the symbols X, x, etc. These must be understood to be present, but are omitted here for the sake of simplicity of expression. We proceed to prove some theorems.

11. *If* x, y, z *have been calculated to the order* $2q-2$, *the constant part in the expansion of* $1/\sqrt{(x^2+y^2+z^2)}$ *can be obtained to the order* $2q$ *without further reference to the equations of motion.*

With the notation just outlined, this theorem may be more fully stated by means of the equation,

$$\left.\begin{aligned} &\text{const. part, order } 2q\text{, in the expansion of } \frac{3}{R_{2q}} \\ &\qquad = \text{const. part, order } 2q\text{, in the expansion of} \\ &\qquad\quad \left[\frac{3}{R_{2q-2}} - \frac{x_0 X_{2q-2} + y_0 Y_{2q-2}}{R_{2q-2}^3} - 3\frac{z_1 z_{2q-2}}{r_0^5}(x_0x_1+y_0y_1)\right]. \end{aligned}\right\} \quad (33)$$

The proof of this is based on Adams' method of proving his theorems referred to below (IV). The equations (10) are, if $r^2 = x^2 + y^2$,

$$\left.\begin{aligned} \ddot{x} - 2n'\dot{y} - 3n'^2x + A'x + B'y &= -\frac{\mu x}{(r^2+z^2)^{\frac{3}{2}}}, \\ \ddot{y} + 2n'\dot{x} \qquad\qquad + B'x + C'y &= -\frac{\mu y}{(r^2+z^2)^{\frac{3}{2}}}, \\ \ddot{z} \qquad + n'^2z + n'^2\rho z \qquad\qquad &= -\frac{\mu z}{(r^2+z^2)^{\frac{3}{2}}}. \end{aligned}\right\} \quad (34)$$

Let x_0, y_0, z_0 be the values of x, y, z, expressed by the same formulæ and with the same angles, but with different arbitrary constants of eccentricity and inclination (i. e. different values of e, γ). We have then

$$\left.\begin{aligned} \ddot{x}_0 - 2n'\dot{y}_0 - 3n'^2x_0 + A'x_0 + B'y_0 &= -\frac{\mu x_0}{(r_0^2+z_0^2)^{\frac{3}{2}}}, \\ \ddot{y}_0 + 2n'\dot{x}_0 \qquad\qquad + B'x_0 + C'y_0 &= -\frac{\mu y_0}{(r_0^2+z_0^2)^{\frac{3}{2}}}, \\ \ddot{z}_0 \qquad + n'^2z_0 + n'^2\rho z_0 \qquad\qquad &= -\frac{\mu z_0}{(r_0^2+z_0^2)^{\frac{3}{2}}}. \end{aligned}\right\} \quad (35)$$

Let the suffix 0 correspond to zero values of e, γ. We suppose as before that expansion may be made in powers of e, γ. Since equations (34) hold generally, they must hold if, after the right-hand sides have been expanded, we stop at any given order in e or γ in all terms. They will therefore hold if we stop at the order $2q$ in e, γ. Hence we can put $x = X_{2q}$, $y = Y_{2q}$, $z = Z_{2q-1}$, as Z^2_{2q-1} is the value of z correct to the order $2q$. Multiply the first two of equations (34) by x_0, y_0 respectively and subtract the sum of the results from the sum of the first two of equations (35) multiplied by X_{2q}, Y_{2q} respectively. We can put the resulting equation into the form

$$\frac{d}{dt}\left[\dot{X}_{2q} x_0 + \dot{Y}_{2q} y_0 - \dot{x}_0 X_{2q} - \dot{y}_0 Y_{2q} - 2n'(Y_{2q} x_0 - X_{2q} y_0)\right]$$
$$= -\mu(X_{2q} x_0 + Y_{2q} y_0)\left(\frac{1}{R^3_{2q}} - \frac{1}{R^3_0}\right).$$

Since the expressions of the coordinates contain t only under the signs, sine cosine, we deduce immediately,
const. part to the order $2q$ inclusive, in the expansion of

$$(X_{2q} x_0 + Y_{2q} y_0)\left(\frac{1}{R^3_{2q}} - \frac{1}{R^3_0}\right) = 0. \qquad (36)$$

Let the notation $C_{2q}[Q]$ denote "the constant part, order $2q$, in the expansion of Q" according to any parameter—which we can put into evidence when it is necessary for the sake of clearness. The equation (36) then gives

$$C_{2q}\left[\frac{X_{2q} x_0 + Y_{2q} y_0}{R^3_{2q}}\right] = C_{2q}\left[\frac{X_{2q} x_0 + Y_{2q} y_0}{R^3_0}\right]$$
$$= C_{2q}\left[\frac{x_{2q} x_0 + y_{2q} y_0}{r^3_0}\right], \qquad (37)$$

since the parts of X_{2q}, Y_{2q} of order $2q$ are x_{2q}, y_{2q} and $R^2_0 = X^2_0 + Y^2_0 = x^2_0 + y^2_0 = r^2_0$. The equation (37) holds for any positive integral value of q.

Now, if we stop at the order $2q$,

$$R^2_{2q} = R^2_{2q-1} + 2x_0 x_{2q} + 2y_0 y_{2q} + 2z_1 z_{2q-1},$$

and therefore

$$C_{2q}\left[\frac{x_0 X_{2q} + y_0 Y_{2q}}{R^3_{2q}}\right] = C_{2q}\left[\frac{x_0 X_{2q-1} + y_0 Y_{2q-1}}{R^3_{2q-1}} + \frac{x_0 x_{2q} + y_0 y_{2q}}{r^3_0}\right.$$
$$\left. - 3(x^2_0 + y^2_0)\frac{x_0 x_{2q} + y_0 y_{2q} + z_1 z_{2q-1}}{r^5_0}\right]. \qquad (38)$$

Putting r_0^2 for $x_0^2 + y_0^2$, we deduce from (37) and (38)

$$C_{2q}\left[\frac{x_0 X_{2q-1} + y_0 Y_{2q-1}}{R_{2q-1}^3}\right] = C_{2q}\left[3\,\frac{x_0 x_{2q} + y_0 y_{2q} + z_1 z_{2q-1}}{r_0^3}\right]. \tag{39}$$

Again, by expansion,

$$C_{2q}\left[\frac{1}{R_{2q}}\right] = C_{2q}\left[\frac{1}{R_{2q-1}} - \frac{x_0 x_{2q} + y_0 y_{2q} + z_1 z_{2q-1}}{r_0^3}\right].$$

Hence by (39),

$$\begin{aligned} C\ \left[\frac{3}{R_{2q}}\right] &= C_{\,q}\left[\frac{3}{R_{2q-1}} - \frac{x_0 X_{2q-1} + y_0 Y_{2q-1}}{R_{2q-1}^3}\right] \\ &= C_{2q}\left[\frac{3}{(R_{2q-2}^2 + 2S_{0,\,2q-1} + 2S_{1,\,2q-1})^{\frac{1}{2}}}\right. \\ &\qquad\qquad \left. - \frac{x_0 X_{2q-2} + y_0 Y_{2q-2} + S_{0,\,2q-1} - z_1 z_{2q-2}}{(R_{2q-2}^2 + 2S_{0,\,2q-1} + 2S_{1,\,2q-1})^{\frac{3}{2}}}\right], \end{aligned}$$

where we have put

$$\begin{aligned} S_{0,\,2q-1} &= x_0 x_{2q-1} + y_0 y_{2q-1} + z_1 z_{2q-2}, \\ S_{1,\,2q-1} &= x_1 x_{2q-1} + y_1 y_{2q-1} + z_1 z_{2q-2}. \end{aligned}$$

Hence by expansion we obtain

$$\left.\begin{aligned} C_{2q}\left[\frac{3}{R_{2q}}\right] = C_{2q}\Big[&\frac{3}{R_{2q-2}} + 9\,\frac{S_{0,\,1}S_{0,\,2q-1}}{r_0^5} - 3\,\frac{S_{1,\,2q-1}}{r_0^3} - \frac{x_0 X_{2q-2} + y_0 Y_{2q-2}}{R_{2q-2}^3} \\ &+ 3S_{0,\,1}\,\frac{S_{0,\,2q-1} - z_1 z_{2q-2}}{r_0^5} \\ &- (r_0^2 + S_{0,\,1})\left(-3\,\frac{S_{0,\,2q-1}}{r_0^5} + 15\,\frac{S_{0,\,2q-1}S_{0,\,1}}{r_0^7} - 3\,\frac{S_{1,\,2q-1}}{r_0^5}\right)\Big] \\ = C_{2q}\Big[&\frac{3}{R_{2q-2}} - \frac{x_0 X_{2q-2} + y_0 Y_{2q-2}}{R_{2q-2}^3} - 3\,\frac{z_1 z_{2q-2} S_{0,\,1}}{r_0^5}\Big)\Big], \end{aligned}\right\} \tag{33'}$$

which formula, since $z_0 = 0$, proves the theorem.

12. *Adams' First Theorem.*

We can deduce this theorem immediately from the above result. It states *that in the constant part of the Moon's Parallax all the terms of orders e^2 and γ^2 are zero.*

In equation (33′) let $q = 1$. We obtain, since $z_0 = 0$,

$$C_2\left[\frac{3}{R_2}\right] = C_2\left[\frac{3}{R_0} - \frac{x_0^2 + y_0^2}{r_0^3}\right] = C_2\left[\frac{2}{r_0}\right] = 0.$$

First, let $\gamma = 0$; this shows that the term in e^2 is zero. Next, let $e = 0$, then similarly the term in $\gamma^2 = 0$. Also it is to be remembered that e, γ occur only in the const. part of $1/R$ in even powers. In the above, e'^2 and its powers are included.

13. *Adams' Second Theorem and its Extension.*

This theorem may be stated as follows:

If

$$C_4\left[\frac{1}{R}\right] = Ee^4 + 2Fe^2\gamma^2 + G\gamma^4 \tag{40}$$

and

$$C_2\,[(c)] = He^2 + K\gamma^2,$$
$$C_2\,[(g)] = Me^2 + N\gamma^2,$$

where E, F, G, H, K, M, N are functions of m, e' only, then

$$\frac{H}{K} = \frac{E}{F}, \quad \frac{M}{N} = \frac{F}{G}.$$

The proof of this theorem may be found in Adams' paper already referred to (IV). It is based on the consideration of the orders of certain functions. In the extension of this theorem which follows, I shall prove again Adams' theorem by another method which, though lacking altogether the neatness of his demonstration, has the advantage of putting the connections between the constant part of the Parallax of the Moon and the motions of the Perigee and Node in a rather clearer light.

Extension of Adams' Second Theorem.

Let

$$e^2T_e = 2\,\frac{a_0^2}{\varkappa}\,\Sigma_j\left[\{2j + 1 + m + (c)\}\,(\varepsilon)_j^2 + \{2j - 1 - m + (c)\}(\varepsilon')_{-j}^2\right],$$

$$\gamma^2T_\gamma = 4\,\frac{a_0^2}{\varkappa}\,\Sigma_j\,[2j + (g)](K)_j^2$$

(see equations (17) and (27)), *where T_e, T_γ are expressed in terms of the same eccentricity and latitude constants (which we have called e, γ) as $1/R$, (c), (g); then*

$$\left.\begin{aligned} HT_e &= 6E, & MT_\gamma &= 6F, \\ KT_e &= 6F, & NT_\gamma &= 6G, \end{aligned}\right\} \tag{41}$$

and thence

$$\frac{T_e}{T_\gamma} = \frac{M}{K}.$$

As before we omit the brackets which denote the presence of powers of e', for the sake of simplicity of expression: it is understood that they exist in what follows.

We have from equations (17) and (31) in this notation,

$$e^4 HT_e = C_{e^4}\left[\frac{x_e X_{e^2} + y_e Y_{e^2}}{R_{e^2}}\right],$$

$$e^2\gamma^2 KT_e = C_{e^2\gamma^2}\left[\frac{x_e (X_e + x_{\gamma^2}) + y_e (Y_e + y_{\gamma^2})}{\{(X_e + x_{\gamma^2})^2 + (Y_e + y_{\gamma^2})^2 + (z_\gamma + z_{e\gamma})^2\}^{\frac{3}{2}}}\right].$$

These can be included in the equation

$$e^4 HT_e + e^2\gamma^2 KT_e = C_{e^4,\ e^2\gamma^2}\left[\frac{x_1 X_2 + y_1 Y_2}{(R_2^2 + 2z_1 z_2)^{\frac{3}{2}}}\right]. \tag{42}$$

For since x, y contain only even powers of γ, $x_1 = x_e$, $y_1 = y_e$, and when $\gamma = 0$, $X_2 = X_{e^2}$, etc., it therefore includes the first equation. If we want the second equation we neglect powers of e in X, Y above the first, hence $X_2 = X_e + x_{\gamma^2}$, etc., for this case.

Again we have from equations (30) and (27)

$$e^2\gamma^2 MT_\gamma = C_{e^2\gamma^2}\left[\frac{z_\gamma (z_\gamma + z_{e\gamma})}{R^3_{e^2}}\right],$$

$$\gamma^4 NT_\gamma = C_{\gamma^4}\left[\frac{z^2_\gamma}{(R^2_{\gamma^2} + z^2_\gamma)^{\frac{3}{2}}}\right],$$

and these can be likewise included in the equation

$$e^2\gamma^2 MT_\gamma + \gamma^4 NT_\gamma = C_{e^2\gamma^2,\ \gamma^4}\left[\frac{z_1 Z_2}{R_2^3}\right], \tag{43}$$

since z contains only odd powers of γ.

Now, since constant terms contain only even powers of e, γ, we have by addition of (42) and (43),

$$e^4 HT_e + e^2\gamma^2 (KT_e + MT_\gamma) + \gamma^4 NT_\gamma = C_4\left[\frac{x_1 X_2 + y_1 Y_2 + z_1 Z_2}{(R_2^2 + 2z_1 z_2)^{\frac{3}{2}}}\right]. \tag{44}$$

By means of this equation we can reduce the four equations (41) to two, namely,

$$C_4\left[\frac{6}{R_4}\right] = C_4\left[\frac{x_1 X_2 + y_1 Y_2 + z_1 Z_2}{(R_2^2 + 2z_1 z_2)^{\frac{3}{2}}}\right], \tag{A}$$

$$C_{e^2\gamma^2}\left[\frac{x_1 X_2 + y_1 Y_2}{(R_2^2 + 2z_1 z_2)^{\frac{3}{2}}}\right] = C_{e^2\gamma^2}\left[\frac{z_1 Z_2}{R_2^3}\right]. \tag{B}$$

For, suppose (A) and (B) to be proved. In (A) put $\gamma = 0$, then $z = 0$, and by

(44) and (40) the first of the relations (41) holds. Next, suppose $e = 0$, then in a similar manner the last of relations (41) holds. Finally, consider the terms in (A) of the order $e^2\gamma^2$. Owing to (B), the second and third of equations (41) will hold. Hence all that is now necessary is to prove the two equations (A) and (B). Since the methods of proof may be of value in investigating similar relations for higher powers of e, γ in the parallax than the fourth, they are given separate statements.

14. *To prove that*

$$C_4\left[\frac{6}{R_4}\right] = C_4\left[\frac{x_1 X_2 + y_1 Y_2 + z_1 Z_2}{(R_2^2 + 2z_1 z_2)^{\frac{3}{2}}}\right]. \tag{A}$$

Attach to x_1, x_2, y_1, etc., a factor α with an index of the same order in e, γ as the quantities to which they are attached. Thus for x_1 we put αx_1, for y_2, $\alpha^2 y_2$, etc.; α will be omitted where its presence is unnecessary. Let

$$S_{rs} = x_r x_s + y_r y_s .$$

Then $$R_2^2 = r_0^2 + \alpha^2 S_{11} + \alpha^4 S_{22} + 2\alpha S_{01} + 2\alpha^2 S_{02} + 2\alpha^3 S_{12} + \alpha^2 z_1^2 .$$

Hence, reducing to a common denominator, the expression

$$\begin{aligned}
\frac{6}{R_2} - 2\,\frac{x_0 X_2 + y_0 Y_2}{R_2^3} &= \frac{2}{R_2^3}\,(2r_0^2 + 3\alpha^2 S_{11} + 3\alpha^2 z_1^2 + 3\alpha^4 S_{22} + 5\alpha S_{01} + 5\alpha^2 S_{02} + 6\alpha^3 S_{12}) \\
&= \frac{1}{R_2^3}\,(4r_0^2 + 5\alpha^2 S_{11} + 5\alpha^2 z_1^2 + 6\alpha^4 S_{22} + 9\alpha S_{01} + 10\alpha^2 S_{02} + 11\alpha^3 S_{12}) \\
&\qquad + \frac{x_1 X_2 + y_1 Y_2 + z_1^2}{R_2^3} \\
&= \alpha^5 . \frac{4\alpha^7 r_0^2 + 5\alpha^9 S_{11} + 5\alpha^9 z_1^2 + 6\alpha^{11} S_{22} + 9\alpha^8 S_{01} + 10\alpha^9 S_{02} + 11\alpha^{10} S_{12}}{(\alpha^8 r_0^2 + \alpha^{10} S_{11} + \alpha^{10} z_1^2 + \alpha^{12} S_{22} + 2\alpha^9 S_{01} + 2\alpha^{10} S_{02} + 2\alpha^{11} S_{12})^{\frac{3}{2}}} \\
&\qquad + \frac{x_1 \dot{X}_2 + y_1 Y_2 + z_1^2}{R_2^3} \\
&= \alpha^5 \frac{d}{d\alpha}\left(\frac{1}{R\alpha^4}\right) + \frac{x_1 X_2 + y_1 Y_2 + z_1^2}{R_2^3} .
\end{aligned}$$

Now, owing to the definition of α we can consider $1/R_2$ as expansible in powers of α, and it is easily seen that

$$\alpha^5 \frac{d}{d\alpha}\left(\frac{1}{R_2 \alpha^4}\right)$$

contains no term of the fourth order in α. Hence

$$C_4\left[\frac{6}{R_2} - 2\,\frac{x_0X_2 + y_0Y_2}{R_2^3}\right] = C_4\left[\frac{x_1X_2 + y_1Y_2 + z_1^2}{R_2^3}\right]. \tag{45}$$

Let now $q = 2$ in the theorem of Art. 11. Equation (33) becomes

$$C_4\left[\frac{3}{R_4}\right] = C_4\left[\frac{3}{R_2} - \frac{x_0X_2 + y_0Y_2}{R_2^3} - 3\,\frac{z_1z_2S_{01}}{r_0^5}\right].$$

Hence from (45),

$$C_4\left[\frac{6}{R_4}\right] = C_4\left[\frac{x_1X_2 + y_1Y_2 + z_1^2}{R_2^3} - 6\,\frac{z_1z_2S_{01}}{r_0^5}\right]$$

$$= C_4\left[\frac{x_1X_2 + y_1Y_2 + z_1Z_2}{(R_2^2 + 2z_1z_2)^{\frac{3}{2}}}\right],$$

which is the result required.

The proof of equation (B) will be divided into two parts, of which the first contains a method of general application to the terms which involve both the constants e, γ.

15. *Lemma. To prove that*

$$C_{e^2\gamma^2}\left[\frac{x_{\gamma^2}X_{e^2} + y_{\gamma^2}Y_{e^2}}{R_{e^2}} - \frac{x_{e^2}X_{\gamma^2} + y_{e^2}Y_{\gamma^2}}{R_{\gamma^2}}\right] = 0. \tag{46}$$

Let, for the purposes of this proof only,

$$\begin{aligned} \xi &= x_0 + x_e + x_{e^2}, & \xi' &= x_0 + x_{\gamma^2}\quad, \\ \eta &= y_0 + y_e + y_{e^2}, & \eta' &= y_0 + y_{\gamma^2}\quad, \\ r^2 &= \xi^2 + \eta^2\quad, & \zeta' &= z_\gamma\quad, \\ & & r'^2 &= \xi'^2 + \eta'^2 + \zeta'^2. \end{aligned}$$

Then ξ, η contain the terms in x, y, independent of γ, as far as the order e^2, and ξ', η', ζ', r', the terms in x, y, z, r, independent of e, as far as the order γ^2.

The proof of the lemma is not difficult. We proceed as in section 11. Taking the equations of motion (34) and substituting therein for x, y, z first ξ, η, 0 and then ξ', η', ζ', we have

$$\ddot{\xi} - 2n'\dot{\eta} - 3n'^2\xi + A'\xi + B'\eta = -\mu\,\frac{\xi}{r^3},$$

$$\ddot{\eta} + 2n'\dot{\xi} \qquad\quad + B'\xi + C'\eta = -\mu\,\frac{\eta}{r^3},$$

and
$$\ddot{\xi}' - 2n'\dot{\eta}' - 3n'^2\xi' + A'\xi' + B'\eta' = -\mu\frac{\xi'}{r'^3},$$
$$\ddot{\eta}' + 2n'\dot{\xi}' \qquad + B'\xi' + C'\eta' = -\mu\frac{\eta'}{r'^3};$$

the right-hand sides of the first pair being expanded as far as the order e^2 and those of the second pair as far as the order γ^2. Take the sum of the first pair multiplied respectively by ξ', η' from the sum of the second pair multiplied respectively by ξ, η. The terms on the left-hand side of the resulting equation form a perfect differential, and therefore contain no constant part. Hence those on the right-hand side as far as the orders e^2, γ^2 contain no constant part. We have then in particular

$$C_{e^2\gamma^2}\left[(\xi\xi' + \eta\eta')\left(\frac{1}{r^3} - \frac{1}{r'^3}\right)\right] = 0.$$

Since ρ contains no terms in γ^2 and ρ' contains no terms in e^2, we have

$$C_{e^2\gamma^2}\left[\frac{\xi\xi' + \eta\eta'}{r^3} - \frac{\xi\xi' + \eta\eta'}{r'^3}\right] = 0,$$

whence follows immediately the equation (46).

From this result we can deduce the result (B). This seems to be most quickly proved by simple expansion.

16. *To prove that*

$$C_{e^2\gamma^2}\left[\frac{x_1X_2 + y_1Y_2}{(R_2^2 + 2z_1z_2)^{\frac{3}{2}}}\right] = C_{e^2\gamma^2}\left[\frac{z_1Z_2}{R_2^3}\right].$$

Let
$$\begin{aligned} S_{01} &= x_0x_e + y_0y_e, & S_{02'} &= x_0x_{\gamma^2} + y_0y_{\gamma^2}, \\ S_{02} &= x_0x_{e^2} + y_0y_{e^2}, & S_{12'} &= x_ex_{\gamma^2} + y_ey_{\gamma^2}, \\ S_{11} &= x_e^2 + y_e^2, & S_{22'} &= x_{e^2}x_{\gamma^2} + y_{e^2}y_{\gamma^2}. \end{aligned}$$

The orders of any terms will then be denoted by the suffixes of S. We have, with this notation, to the order required,

$$R_{e^2}^2 = r_0^2 + S_{11} + 2S_{01} + 2S_{02},$$
$$R_{\gamma^2}^2 = r_0^2 + 2S_{02'} + z_1^2,$$

and therefore, by expansion of equation (46),

$$C_{e^2\gamma^2}\left[\frac{S_{22'}}{r_0^3} - 3\frac{S_{01}S_{12'}}{r_0^5} + S_{02'}\left\{-3\frac{S_{02} + \frac{1}{2}S_{11}}{r_0^5} + \frac{15}{2}\frac{S_{01}^2}{r_0^7}\right\}\right.$$
$$\left. - \frac{S_{22'}}{r_0^3} + 3S_{02}\frac{S_{02'} + \frac{1}{2}z_1^2}{r_0^5}\right] = 0,$$

or, after reduction and multiplication by 2,

$$C_{e^2\gamma^2}\left[-6\frac{S_{01}S_{12'}}{r_0^5}-3\frac{S_{02'}S_{11}}{r_0^5}+15\frac{S_{01}^2S_{02'}}{r_0^7}+3\frac{S_{02}z_1^2}{r_0^5}\right]=0.$$

Rearranging, this equation may be written

$$C_{e^2\gamma^2}\left[S_{01}\left\{-3\frac{S_{12'}}{r_0^5}+15\frac{S_{01}S_{02'}+\frac{1}{2}z_1^2S_{01}}{r_0^7}\right\}\right.$$
$$\left.-3S_{11}\frac{S_{02'}+\frac{1}{2}z_1^2}{r_0^5}-3S_{12'}\frac{S_{01}}{r_0^5}-z_1^2\left\{-3\frac{S_{02}+\frac{1}{2}S_{11}}{r_0^5}+\frac{15}{2}\frac{S_{01}^2}{r_0^7}\right\}\right]=0,$$

which is the expansion of

$$C_{e^2\gamma^2}\left[\frac{S_{01}+S_{11}+S_{12'}-z_1^2}{(r_0^2+S_{11}+2S_{01}+2S_{02}+2S_{02'}+2S_{12'}+z_1^2)^{\frac{3}{2}}}\right]=0.$$

Referring to the notation defined by equations (32), this is evidently the same, if we omit superfluous terms, as

$$C_{e^2\gamma^2}\left[\frac{x_1X_2+y_1Y_2-z_1^2}{R_2^3}\right]=0.$$

This equation is easily shown to be the same as

$$C_{e^2\gamma^2}\left[\frac{x_1X_2+y_1Y_2-z_1Z_2}{(R_2^2+2z_1z_2)^{\frac{3}{2}}}\right]=0,$$

whence follows the equation (B).

The above results give some insight into the nature of the connection between the constant parts of the Moon's Parallax and the motions of its Perigee and Node. If we had assumed the results of Adams' second theorem (which is here a deduction from the equations (A) and (B)), it would have been unnecessary to prove (B). But it is desirable to see exactly by what means the results come, and thence to get an idea as to the way in which they might be extended to higher powers of e, γ. The methods followed above appear to be sufficient to indicate further relations if they exist. The chief trouble is to investigate the terms depending on both e, γ, those depending on one only of these constants requiring much less labor. The results are independent of the particular method of defining e, γ, so long as these quantities are of the same order as the coefficients of the principal elliptic term in longitude and of the principal term in latitude respectively. In fact, their definition is restricted only by the assumption that the coordinates are expressible by means of the same angles at any stage of the approximations. We simply approximate to the coefficients. A reference to Adams' paper (IV) will make these points clearer.

17. *Extension to Terms of Higher Orders.*

It is not difficult to extend these formulæ when terms of higher orders in the expressions of c, g are under consideration. But simple relations with the constant part of the Parallax of the Moon, like those which exist for $q = 1$ and $q = 2$, do not seem to be present in the terms of higher orders. The methods used here should give any relations which may exist, and it is not at all improbable that there are such.

The connections between the motions of Perigee and Node and the Parallax are really due to the fact that these are all independent (in the sense of the theorem of section 11) of the coefficients of the next higher order than the terms under consideration in the Perigee and Node. The formulæ for the latter, though symmetrical in x, y or z, are not so with reference to the quantities x_0, x_e, x_{γ^2}, etc., while the expression for the Parallax is so. Consequently the results of Adams' theorems and their extensions must be rather considered from this point of view as accidental.

The terms in the motion of the Perigee depend principally on the function

$$\frac{x_1 X_{2q-2} + y_1 Y_{2q-2}}{(R^2_{2q-2} + 2z_1 z_{2q-2})^{\frac{3}{2}}},$$

where $2q - 2$ is the order of the terms required.

As an example, take the case $q = 3$ in the theorem of Art. 11. This gives, if we consider only the constant part of the order e^6, i. e. putting $\gamma = 0$,

$$C_{e^6}\left[\frac{9}{R_6}\right] = C_{e^6}\left[\frac{9}{R_4} - 3\,\frac{x_0 X_4 + y_0 Y_4}{R_4^3}\right],$$

which may be shown to be equal to

$$C_{e^6}\left[\tfrac{1}{2}a^7 \frac{d}{da}\left(\frac{1}{Ra^6}\right) + 2\,\frac{x_1 X_4 + y_1 Y_4}{R_4^3} + \frac{x_2 X_3 + y_2 Y_3}{R_4^3} - \frac{x_4 X_1 + y_4 Y_1}{R_4^3}\right],$$

after the method and notation used before. The first term on the right-hand side vanishes, having no coefficient of the order e^6; the second term is that required for the determination, in the motion of the Perigee, of the terms of order

e^4 [equation (18)], but the other two terms do not vanish. Some reductions are possible in these latter terms, but x_4 and y_4 cannot be eliminated.

PART III.

TERMS OF LONG AND SHORT PERIODS WHOSE COEFFICIENTS ARE LARGE IN COMPARISON TO THE ORDERS OF THE COEFFICIENTS.

There are some results which can be deduced from the infinite determinant which are of importance when short-period terms are under discussion.

Let the arguments of any series of terms in x, y differing by multiples of $2\nu(t - t_0)$ be

$$(2j + \Lambda)\nu t + \text{const.},$$

where Λ is of the form $j_1 + km + pc + 2qg$, and consequently the coefficients of the order $a'^{-j'} e'^{k'} e^{p'} \gamma^{2q'}$. Here k, p, q are integers positive or negative. Also $j_1 = 0$ when j' is even and $j_1 = 1$ when j' is odd; in addition, we have

$$k' = (k) \text{ or } k' = (k) + \text{even positive integer, etc.}$$

Let the coefficients corresponding to any given value of Λ be λ_j, λ'_j. The equations which determine them are

$$(2j+1+m+\Lambda)^2\lambda_j + \Sigma_i M_i \lambda_{j-i} + \Sigma_i N_i \lambda'_{i-j} = \text{known terms independent of } \lambda, \lambda',$$
$$(2j-1-m+\Lambda)^2\lambda'_{-j} + \Sigma_i M_{-i} \lambda'_{i-j} + \Sigma_i N_{-i} \lambda'_{j-i} = \text{`` \quad `` \quad `` \quad ``}$$

Suppose that the solution of these equations is found by means of determinants. A reference to equations (14) shows that the common denominator in the expressions of all the coefficients λ_j, λ'_j is $\Delta(\Lambda)$ or $\nabla(\Lambda)$. Now the identity

$$\nabla(c) \equiv (\cos \pi c - \cos \pi c_1)(\cos \pi c - 1)$$

of Art. 4 holds for all values of c, and therefore when we put Λ for c. Hence, dropping the suffix of c_1 so that c denotes the principal part of the motion of Perigee, we have for the common denominator $\nabla(\Lambda)$ the value

$$(\cos \pi\Lambda - \cos \pi c)(\cos \pi\Lambda - 1).$$

From this we conclude that the coefficients λ_j, λ'_j may become large when either

$$\cos \pi\Lambda - \cos \pi c \quad \text{or} \quad \cos \pi\Lambda - 1$$

is a small quantity. Since c is nearly equal to unity, these expressions cannot become small together. This shows that when we have the equations

$$\Lambda = c + \text{even integer} \quad \text{or} \quad \Lambda = \text{even integer},$$

approximately satisfied, the quantities λ_j, λ'_j may become large. The term "even integer" includes negative as well as positive even integers and also zero.

The second case is that of long-period terms. The result obtained does not add anything to our knowledge of them, except perhaps the fact (which applies also to the terms affected by the first equation) that all terms whose arguments differ from $\Lambda\nu t +$ const. by even multiples of $2\nu(t - t_0)$ should be considered together. The first equation affects only short-period terms: the periods of these terms differ little from those of the principal elliptic inequalities, that is, from $2\pi/(2j \pm c)\nu$. We have neglected in c all terms but those depending on m only, but the results evidently hold when we consider the full value of c. If we had used numerical values for all the constants it would have been necessary to take this full value. In the case before us, where we use the numerical value of m only, we take the part of c depending on m only.

It seems to have been usual to consider short-period terms as those whose periods approximate to that of the mean motion. The result shows that we ought rather to consider the terms whose periods approximate to that of the principal elliptic inequality. The result obtained in a previous investigation (VII, p. 254) is evidently only a first approximation to that obtained here. The difference is due to the fact that here we include at once all the terms whose arguments differ from that considered by even multiples of $2\nu(t - t_0)$, and we therefore include in the coefficients all powers of m.

Only a finite number of the coefficients λ_j, λ'_j will become large relatively to the rest, but the whole series will be large relatively to their order in e, e', γ, $1/a'$. No coefficients of terms of long period become large by the integration of the latitude equation. The same remarks, as to the terms of short period, apply if for "principal elliptic inequalities" we substitute "principal latitude inequalities," i. e. those of period $2\pi/(2j \pm g)\nu$.

For the calculations of Part I, where we are especially concerned, in the infinite determinant, with the values of c, g which depend on m only, the above remarks may be included in the following rule: *When calculating any series of terms in* x, y *in which* Λ *has the value* $i + km + pc + 2qg$, *in order to discover if a large divisor is present it is necessary to examine whether for positive or negative* *integral values of* i, k, p, q *either of the expressions*

$$i + km + (p - 1)c + 2qg, \quad i + km + pc + 2qg$$

becomes small. In calculating terms in z, *we examine whether the second expression may become small.* The last statement arises from the fact that in latitude Λ is of the form

$$i + km + pc + (2q + 1)g,$$

and this for large coefficients must be approximately equal to g. It is to be remembered that the rule applies to all terms wherein the coefficient of νt in the argument differs from Λ by even multiples of ν.

The part of the coefficient of lowest order of a term in x, y whose argument is $(i + km + pc + 2qg)\nu t +$ const., is of order $e'^k e^p \gamma^{2q}$ or $e'^k e^p \gamma^{2q} . 1/a'$, according as i is even or odd: in z the argument being $\{i + km + pc + (2q + 1)\} \nu t +$ const., the orders are for i even or odd, $e'^k e^p \gamma^{2q+1}$ and $e'^k e^p \gamma^{2q+1} 1/a'$ respectively. Hence *if there is a long-period inequality with a large coefficient of order* $e'^k e^p \gamma^{2q}$ *in* x, y *there is a short-period inequality with a large coefficient of order* $e'^k e^{p+1} \gamma^{2q}$ *in* x, y *and a similar one of order* $e'^k e^p \gamma^{2q+1}$ *in* z. In the indices of e, e', γ the numbers k, p, q receive positive values.

By considering the algebraical values of the principal parts of c, g we can also arrive at the order of the denominator in relation to m, and thence, by looking at the manner in which the denominators are built up, the orders of any terms, with respect to the parameter m, can be obtained.

Christ's College, Cambridge, *December 24th*, 1894.

Ueber den Sinn der Windung in den singulären Puncten einer Raumcurve.

Von Otto Staude *in Rostock.*

Die analytische Unterscheidung rechts und links gewundener Elemente einer Raumcurve ist kürzlich von Herrn Kneser[*] eingehend untersucht worden, soweit es sich um reguläre Puncte der Curve handelt. Die vorliegende Mittheilung dehnt diese Untersuchung auf singuläre Puncte aus.

Die Schwierigkeit liegt dabei in der Durchführung gewisser Vorzeichenbestimmungen, für welche indessen ein einfacher Grundgedanke maassgebend ist. Wenn nämlich bei der Untersuchung der Bahncurve eines bewegten Punctes in einer geraden Linie, in einer Ebene oder im Raume neben anderen Bestimmungsstücken das doppelte Vorzeichen benutzt wird, so kann dies nach folgender Stufenfolge geschehen. In der *Geraden* entscheidet das Vorzeichen über die beiden entgegengesetzten Richtungen, welche die Tangente der Bahncurve haben kann oder, anders ausgedrückt, über den *Sinn der Linearfortschreitung*. In der *Ebene*, wo die ∞^1 möglichen Richtungen der Tangente der Bahncurve durch einen Winkel (2 Richtungscosinus) bestimmt werden, entscheidet das Vorzeichen über die beiden bei gegebener Tangente noch möglichen Richtungen der Normale oder, was auf dasselbe hinauskommt, über den *Sinn der Krümmung*. Im *Raume*, wo die ∞^2 möglichen Richtungen der Tangente der Bahncurve durch zwei Winkel (3 Richtungscosinus) und die bei gegebener Tangente möglichen ∞^1 Richtungen der Hauptnormale durch einen Winkel bestimmt werden, entscheidet das Vorzeichen über die beiden bei gegebener Tangente und Hauptnormale noch möglichen Richtungen der Binormale oder über den *Sinn der Windung* (Torsion).

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[*] **Bemerkungen über die Frenet-Serret'schen Formeln etc., Journal für die reine und angew. Mathematik, Bd. 113, S. 89.**

Dieses Princip ist im Folgenden überall festgehalten und mit den Bedingungen der singulären Puncte verknüpft worden. Nach den einleitenden Festsetzungen des §1 und §2 und der Eintheilung der singulären Puncte in §3, werden zunächst in §4 für die singulären Puncte charakterische Coordinatensysteme eingeführt, welche der weiteren Discussion als Grundlage dienen. Hierauf wird in §5 die Bedeutung der Vorzeichen der aus den Differentialquotienten der 3 Coordinaten des Curvenpunctes nach dem Parameter gebildeten dreireihigen Determinanten entwickelt, deren erste, nur 1., 2. und 3. Differentialquotienten enthaltende, in gleichem Sinne von Kneser* betrachtet und deren Verschwinden als Kriterium der singulären Puncte überhaupt von Fine† untersucht worden ist. In §6 endlich werden die Hauptresultate in einer Tabelle zusammengefasst und weiter erläutert.

Obwohl die angewandte Methode bei n-Dimensionen und bei beliebiger Ordnung der singulären Puncte sich durchführen lässt, beschränkt sich die gegenwärtige Mittheilung auf die zur Charakterisirung der Methode ausreichenden 8 Fälle des gewöhnlichen Raumes, welche nach v. Staudt's Vorgange von Ch. Wiener‡ besprochen und in Modellen|| zur Darstellung gebracht worden sind.

§1. *Festsetzung über den Sinn eines Axensystems, einer Drehung und einer Windung.*

Um den Anfangspunct O eines rechtwinkligen Axensystems $Oxyz$ im Raume denken wir uns eine Kugel beschrieben und den Umfang des sphärischen Dreiecks, welches die 3 positiven Halbaxen Ox, Oy, Oz auf der Aussenfläche der Kugel bestimmen, in der Reihenfolge x, y, z der 3 Ecken durchlaufen. Jenachdem diese Durchlaufung dem Drehungssinn des Uhrzeigers entgegengesetzt oder mit ihm übereinstimmend ist, nennen wir das Axensystem $Oxyz$ positiv oder negativ orientirt. Wir benutzen in der Folge stets ein *positiv orientirtes Axensystem Oxyz* als Coordinatensystem.§

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* a. a. O. S. 96.

† On the Singularities of Curves of Double Curvature, American Journal of Mathematics, Vol. VIII, S. 175.

‡ Die Abhängigkeit der Rückkehrelemente der Projectionen einer unebenen Curve von denen der Curve selbst, Schlömilch's Zeitschrift, Bd. 25, S. 95.

|| Acht Modelle über die Abhängigkeit der Rückkehrelemente etc., Verlag von L. Brill in Darmstadt.

§ Übereinstimmend mit Kneser, a. a. O., S. 91.

In Bezug auf ein solches sei (vergl. Fig. 1a) $P = x, y, z$ ein laufender Punct und $Q = x, y$ seine Projection auf die xy-Ebene. Ferner sei t der Winkel des Strahles OQ gegen die positive x-Axe und zwar wachse derselbe, von der positiven z-Axe aus gesehen, im *positiven Drehungssinne*, welchen wir stets dem des Uhrzeigers entgegengesetzt annehmen.

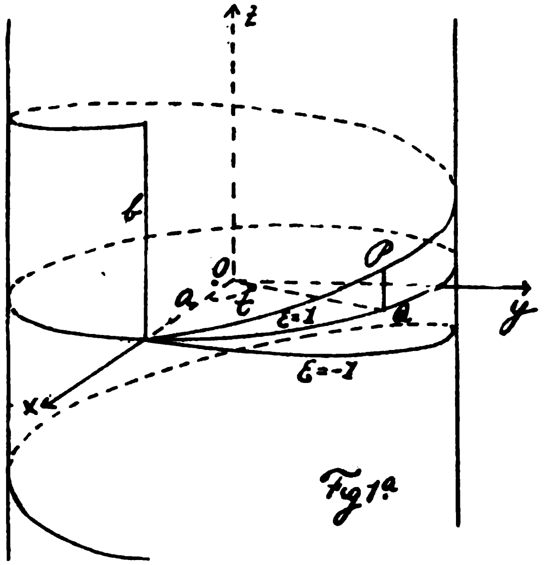

Fig. 1a

Alsdann stellen die Gleichungen:

$$x = a\cos t,\quad y = a\sin t,\quad z = \varepsilon\frac{bt}{2\pi},\qquad \varepsilon = \pm 1 \tag{1}$$

eine Schraubenlinie vom Radius a und der Ganghöhe b dar, welche positiv oder negativ gewunden ist, jenachdem $\varepsilon = +1$ oder $\varepsilon = -1$ genommen wird. Um nämlich den Sinn der Windung zu definiren, betrachten wir die Schraubenlinie in der Richtung ihrer Axe und lassen sie von einem Puncte in solcher Weise durchlaufen, dass uns seine drehende Bewegung um die Axe positiv erscheint: jenachdem dann seine fortschreitende Bewegung längs der Axe auf uns zu oder von uns wegführt, nennen wir die Schraubenlinie *positiv oder negativ gewunden.** Lassen wir nun in (1) den Parameter t wachsen, so bewegt sich der laufende Punct P von der positiven z-Axe aus gesehen (vgl. Fig. 1a) im positiven Drehungssinne, während gleichzeitig z wächst für $\varepsilon = 1$ und abnimmt für $\varepsilon = -1$, so dass die über die Schraubenlinie (1) gemachte Angabe unserer Definition entspricht. Bei abnehmendem t würden wir, von der negativen z-Axe aus zusehend, den gleichen Sinn der Windung finden.

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* Wie bei Kneser, a. a. O., S. 100.

§ 2. *Die positive Durchlaufungsrichtung einer Curve und die Nachbarpuncte eines Punctes.*

Durch die Gleichungen:

$$x = \phi(t), \quad y = \psi(t), \quad z = \chi(t), \tag{2}$$

bezogen auf das in § 1 eingeführte Coordinatensystem $Oxyz$, sei eine Raumcurve gegeben. Den Parameter t, der für die ganze Betrachtung derselbe bleiben soll, denken wir uns etwa als Zeit gedeutet und die Curve (2) dem entsprechend als Bahncurve eines bewegten Punctes x, y, z. Als *positive Durchlaufungsrichtung der Curve* im Bereiche eines Punctes P derselben bezeichnen wir diejenige, welche einer wachsenden Bewegung des Parameters t, bezüglich der fortschreitenden Zeit entspricht.

Wir setzen voraus, dass im Bereiche des Punctes P, also für hinreichend kleine absolute Werthe der Grösse τ die Taylor'sche Entwicklung:

$$\begin{aligned} \phi(t+\tau) &= \phi(t) + \phi'(t).\tau + \tfrac{1}{2}\phi''(t).\tau^2 + \dots. \\ &= x + x'\tau + \tfrac{1}{2}x''\tau^2 + \dots., \end{aligned} \tag{3}$$

sowie die entsprechenden für y und z gelten. Darnach betrachten wir eine Reihe in der positiven Durchlaufungsrichtung auf P folgender Nachbarpuncte $P_1, P_2, P_3, \dots.$ mit den Coordinaten:

$$\left.\begin{aligned} &x_1 = x + x'\tau_1, \quad y_1 = y + y'\tau_1, \quad z_1 = z + z'\tau_1, \\ &x_2 = x + x'\tau_2 + \tfrac{1}{2}x''\tau_2^2, \quad y_2 = y + y'\tau_2 + \tfrac{1}{2}y''\tau_2^2, \dots. \\ &x_3 = x + x'\tau_3 + \tfrac{1}{2}x''\tau_3^2 + \tfrac{1}{6}x'''\tau_3^3, \dots., \dots., \\ &\dots\dots\dots\dots\dots\dots\dots\dots \end{aligned}\right\} \tag{4}$$

und ebenso eine Reihe vorangehender Nachbarpuncte $P_{-1}, P_{-2}, P_{-3}, \dots.$ mit den Coordinaten:

$$\left.\begin{aligned} &x_{-1} = x - x'\tau_1, \quad y_{-1} = y - y'\tau_1, \quad z_{-1} = z - z'\tau_1, \\ &x_{-2} = x - x'\tau_2 + \tfrac{1}{2}x''\tau_2^2, \quad y_{-2} = y - y'\tau_2 + \tfrac{1}{2}y''\tau_2^2, \dots., \\ &x_{-3} = x - x'\tau_3 + \tfrac{1}{2}x''\tau_3^2 - \tfrac{1}{6}x'''\tau_3^3, \dots\dots\dots \\ &\dots\dots\dots\dots\dots\dots\dots\dots \end{aligned}\right\} \tag{4}$$

Diesen Angaben liegt die Vorstellung zu Grunde, dass bei einem vorgegebenen gemeinsamen Genauigkeitsgrade von den kleinen *positiven Grössen* τ_1, τ_2, τ_3, ... beziehungsweise die 2., 3., 4., ... Potenzen vernachlässigt werden können.

§3. *Classification der Curvenpuncte.*

Mit Rücksicht auf die Lage der Nachbarpuncte P_1, P_2, P_3, ... nehmen wir nun eine Classification der Puncte P der Curve (2) vor. Wir suchen nämlich, von P ausgehend, den nächsten Nachbarpunct $P_i (i = 1, 2, 3, \dots)$, der aus dem Puncte P heraustritt, hierauf den nächsten Nachbarpunct $P_{i+\varkappa} (\varkappa = 1, 2, 3, \dots)$, der aus den Geraden PP_i heraustritt, endlich den nächsten, der aus der Ebene $PP_iP_{i+\varkappa}$ heraustritt. Auf Grund dieses Verfahrens* erhalten wir folgende Eintheilung:

- I. P_1 nicht im Puncte P.
    - A. P_2 nicht in der Geraden PP_1.
        - a. P_3 nicht in der Ebene PP_1P_2;
        - b. P_3 (aber nicht P_4) in der Ebene PP_1P_2.
    - B. P_2 (aber nicht P_3) in der Geraden PP_1.
        - a. P_4 nicht in der Ebene $P_1P_2P_3$;
        - b. P_4 (aber nicht P_5) in der Ebene $P_1P_2P_3$.
- II. P_1 (aber nicht P_2) im Puncte P.
    - A. P_3 nicht in der Geraden P_1P_2.
        - a. P_4 nicht in der Ebene $P_1P_2P_3$;
        - b. P_4 (aber nicht P_5) in der Ebene $P_1P_2P_3$.
    - B. P_3 (aber nicht P_4) in der Geraden P_1P_2.
        - a. P_5 nicht in der Ebene $P_2P_3P_4$;
        - b. P_5 (aber nicht P_6) in der Ebene $P_2P_3P_4$.

(5)

Diese Eintheilung kann nach jedem der 3 Eintheilungsgründe, also in den Richtungen a, b, ...; A, B, ... und I, II, ... weiter fortgesetzt werden. Wir begnügen uns jedoch mit der näheren Betrachtung der in der Tabelle (5) wirklich aufgeführten 8 Fälle, derselben, auf welche sich die oben erwähnten Wiener'schen Modelle beziehen (vgl. unten §6).

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* Nach Fine, a. a. O., §2, S. 160.

§4. *Die charakteristischen Coordinatensysteme.*

In den Fällen I dürfen nach (4) die Grössen x', y', z' für den Punct P nicht alle drei verschwinden. Wir definiren die positive Tangente t (von dem gleichbezeichneten Parameter t in (2) zu unterscheiden) im Puncte P, übereinstimmend mit der positiven Durchlaufungsrichtung der Curve, als die von P nach P_1 hin laufende Gerade. Ihre Richtungscosinus sind nach (4):

$$\alpha = \frac{x'}{s'}, \quad \beta = \frac{y'}{s'}, \quad \gamma = \frac{z'}{s'}, \quad s' = \sqrt{x'^2 + y'^2 + z'^2}, \tag{6}$$

wo die doppelt gestrichene Quadratwurzel immer deren absoluten Werth bezeichnet. Nach (4) liegt P_{-1} ebenfalls auf t und fällt die Richtung von P_{-1} nach P mit der Richtung (6) von t zusammen. Die Puncte P_{-1}, P, P_1 folgen also auf der Tangente in deren positiver Richtung aufeinander:

Der laufende Punct der Curve ändert an der Stelle P den Sinn seiner Linearfortschreitung nicht.

In den Fällen I, A dürfen die Bedingungen $x':y':z' = x'':y'':z''$ nicht erfüllt sein. Die Schmiegungsebene Σ ist als Verbindungsebene der Puncte P, P_1, P_2 bestimmt und mit ihr Hauptnormale h ($\perp t$ in Σ) und Binormale b ($\perp \Sigma$), beide zunächst ohne Pfeilspitzen. Wir bestimmen die letzteren aus den beiden Bedingungen, dass erstens das vom Puncte P ausgehende Coordinatensystem $\xi = t$, $\eta = h$, $\zeta = b$ positiv orientirt und zweitens P_2 auf der positiven Seite der Ebene bt, d. h. auf der Seite der positiven Hauptnormale h gelegen sei. Aus diesen Bedingungen ergeben sich als Richtungscosinus der positiven Hauptnormale:

$$\alpha^{\cdot} = \rho\,\frac{s'x'' - x's''}{s'^3}, \quad \beta^{\cdot} = \rho\,\frac{s'y'' - y's''}{s'^3}, \quad \gamma^{\cdot} = \rho\,\frac{s'z'' - z's''}{s'^3} \tag{7}$$

und der positiven Binormale:

$$\alpha^{\cdot\cdot} = \rho\,\frac{y'z'' - z'y''}{s'^3}, \quad \beta^{\cdot\cdot} = \rho\,\frac{z'x'' - x'z''}{s'^3}, \quad \gamma^{\cdot\cdot} = \rho\,\frac{x'y'' - y'x''}{s'^3}, \tag{7}$$

wobei die positive Grösse ρ durch die Gleichung

$$\frac{s'^3}{\rho} = \sqrt{(y'z'' - z'y'')^2 + (z'x'' - x'z'')^2 + (x'y'' - y'x'')^2} = s'\sqrt{x''^2 + y''^2 + z''^2 - s''^2},$$

definirt ist. Denn mit den in (6) und (7) angegebenen Richtungscosinus ist erstens die Determinante:

$$\begin{vmatrix} \alpha & \alpha^{\cdot} & \alpha^{\cdot\cdot} \\ \beta & \beta^{\cdot} & \beta^{\cdot\cdot} \\ \gamma & \gamma^{\cdot} & \gamma^{\cdot\cdot} \end{vmatrix} = +1$$

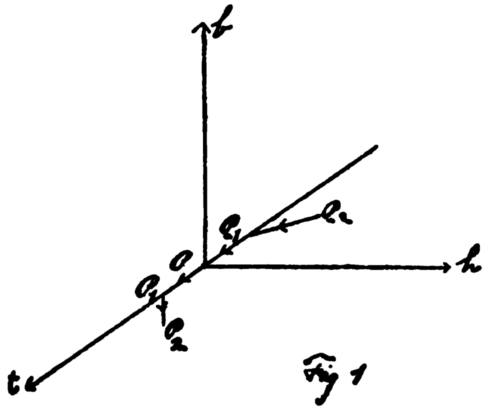
und zweitens die η-Coordinate des Punctes P_2:

$$\eta_2 = \alpha^{\cdot}(x_2 - x) + \beta^{\cdot}(y_2 - y) + \gamma^{\cdot}(z_2 - z) = \tfrac{1}{2}\,\frac{s'^2}{\rho}\,\tau_2^2$$

positiv. Beiläufig ergiebt sich, dass auch der Punct P_{-2} mit:

$$\eta_{-2} = \tfrac{1}{2}\,\frac{s'^2}{\rho}\,\tau_2^2$$

auf der positiven Seite der Ebene $\zeta\xi$ liegt.



Das durch (6) *und* (7) *bestimmte Coordinatensystem thb* (*vgl. Fig.* 1) *ist allen Fällen I, A charakteristisch.*

(Die Figur stellt nur schematisch die Lage der Axen gegen die Curvenpuncte dar und giebt nicht das thatsächliche Längenverhältniss der Strecken PP_1 und P_1P_2 (vgl. §2) wieder.)

In den Fällen I, B sind die Bedingungen $x' : y' : z' = x'' : y'' : z''$ erfüllt, womit die Puncte $P_{\pm 2}$ in die Tangente t zu liegen kommen. Die Formeln (6) behalten ihre Gültigkeit, können aber jetzt noch in anderer Weise dargestellt werden. Es sei nämlich, zufolge der Voraussetzung, mit einem Proportionalitätsfactor εx^2 $(\varepsilon = \pm 1)$:

$$x'' = \varepsilon x^2 x', \quad y'' = \varepsilon x^2 y', \quad z'' = \varepsilon x^2 z'. \tag{8}$$

Setzen wir diese Werthe in die aus der quadrirten letzten Gleichung (6) durch Differentiation nach t folgende Gleichung:

$$s's'' = x'x'' + y'y'' + z'z'' \tag{6'}$$

ein, so erhalten wir mit Weglassung des Factors s':

$$s'' = \varepsilon\varkappa^2 s' \tag{8}$$

und damit für die Gleichungen (6) die aequivalente Form:

$$\alpha = \frac{x''}{s''}, \quad \beta = \frac{y''}{s''}, \quad \gamma = \frac{z''}{s''}, \quad s'' = \varepsilon\sqrt{x''^2 + y''^2 + z''^2}. \tag{9}$$

Zugleich reducirt sich die aus (6′) folgende Gleichung:

$$s's''' + s''^2 = x'x''' + x''^2 + y'y''' + y''^2 + z'z''' + z''^2 \tag{6''}$$

nach (8) und (9) auf:

$$s''s''' = x''x''' + y''y''' + z''z'''. \tag{8'}$$

Während nun die Schmiegungsebene Σ im Puncte P_1 mit der Gleichung:

$$(X - x)(y'z'' - z'y'') + (Y - y)(z'x'' - x'z'') + (Z - z)(x'y'' - y'x'') = \Sigma(t) = 0$$

in laufenden Coordinaten X, Y, Z in Bezug auf das Coordinatensystem $Oxyz$, unbestimmt und die Formeln (7) unbrauchbar werden, sind die vereinigten Schmiegungsebenen $\Sigma_{\pm 1}$ der Puncte $P_{\pm 1}$ als Verbindungsebenen der Puncte P_1, P_2, P_3 und P_{-1}, P_{-2}, P_{-3} gegeben oder durch die Gleichung:

$$\Sigma(t) \pm \Sigma'(t) . \tau_1 = 0,$$

die sich auf:

$$(X - x)(y'z''' - z'y''') + \ldots . + \ldots . = 0 \text{ oder } (X - x)(y''z''' - z''y''') + \ldots . + \ldots . = 0$$

reducirt. Zugleich sind die Hauptnormalen $h_{\pm 1}$ ($\perp t$ in $\Sigma_{\pm 1}$) und Binormalen $b_{\pm 1}$ ($\perp \Sigma_{\pm 1}$) bis auf die Pfeilspitzen bestimmt. Die letzteren definiren wir durch die beiden Bedingungen, dass erstens jedes der beiden Coordinatensysteme $\zeta_\pm = t$, $\eta_\pm = h_{\pm 1}$, $\zeta_\pm = b_{\pm 1}$ positiv orientirt und zweitens P_3 auf der positiven Seite der Ebene $b_1 t$ und P_{-3} auf der positiven Seite der Ebene $b_{-1} t$ gelegen sei. Aus diesen Bedingungen ergeben sich mit Rücksicht auf (8′) als Richtungscosinus der positiven Hauptnormalen $h_{\pm 1}$:

$$\alpha_{\pm 1} = \pm\lambda \frac{s''x''' - x''s'''}{s''^2}, \quad \beta_{\pm 1} = \pm\lambda \frac{s''y''' - y''s'''}{s''^2}, \quad \gamma_{\pm 1} = \pm\lambda \frac{s''z''' - z''s'''}{s''^2} \tag{10}$$

und der positiven Binormalen $b_{\pm 1}$:

$$\alpha^{\cdot\cdot}_{\pm 1} = \pm\lambda\frac{y'z''' - z'y'''}{s''^3},\quad \beta^{\cdot\cdot}_{\pm 1} = \pm\lambda\frac{z'x''' - x'z'''}{s''^3},\quad \gamma^{\cdot\cdot}_{\pm 1} = \pm\lambda\frac{x'y''' - y'x'''}{s''^3}, \quad (10)$$

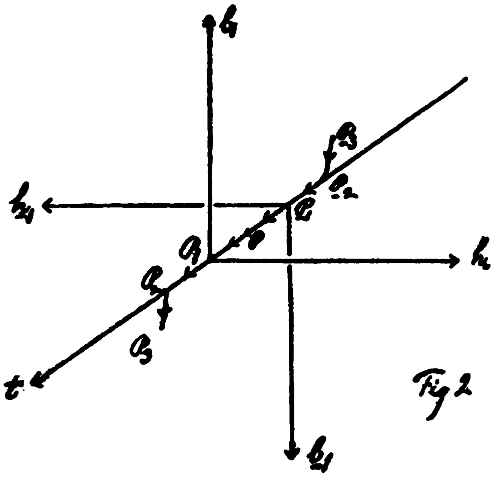
wobei die positive Grösse λ durch die Gleichung:

$$\frac{\varepsilon s''^3}{\lambda} = \sqrt{(y'z''-z'y'')^2+(z'x''-x'z'')^2+(x'y''-y'x'')^2} = \varepsilon s''\sqrt{x'''^2+y'''^2+z'''^2-s'''^2}$$

definirt ist und, *wie auch im Folgenden immer, je alle oberen und alle unteren Vorzeichen jeder Zeile zusammengehören.* Denn mit den Werthen (9) und (10) werden die beiden Determinanten $|\alpha\beta^{\cdot}_{\pm 1}\gamma^{\cdot\cdot}_{\pm 1}| = +1$ und die zweite Coordinate η_3 des Punctes P_3 im ersten und η_{-3} des Punctes P_{-3} im zweiten der erwähnten Coordinatensysteme:

$$\eta_{\pm 3} = \alpha^{\cdot}_{\pm 1}(x_{\pm 3} - x) + \beta^{\cdot}_{\pm 1}(y_{\pm 3} - y) + \gamma^{\cdot}_{\pm 1}(z_{\pm 3} - z) = \frac{1}{6}\frac{s''^2}{\lambda}\tau_3^3$$

positiv. Die Gleichungen (10) sind von dem in (8) eingeführten ε unabhängig.



Die durch (9) *und* (10) *bestimmten Coordinatensysteme* th_1b_1 *und* $th_{-1}b_{-1}$ (*vgl. Fig.* 2) *sind allen Fällen I, B charakteristisch.*

In den Fällen II verschwinden die Grössen x', y', z' für den Punct P und wird mit dem Zusammenfall der Puncte P_1 und P die Tangente $t = PP_1$, sowie die Normalebene:

$$(X - x)x' + (Y - y)y' + (Z - z)z' = N(t) = 0$$

unbestimmt. Dagegen haben wir für die vereinigten Normalebenen $N_{\pm 1}$ der Puncte $P_{\pm 1}$:

$$N(t) \pm N'(t)\tau_1 = 0 \text{ oder } (X-x)x'' + \ldots\ldots + \ldots\ldots = 0$$

und, übereinstimmend mit der positiven Durchlaufungsrichtung der Curve, als positive Tangente t_1 in P_1 die Gerade von P_1 nach P_2 und als positive Tangente t_{-1} in P_{-1} die Gerade von P_{-2} nach P_{-1}. Ihre Richtungscosinus sind nach (4):

$$\alpha_{\pm 1} = \pm \frac{x''}{s''},\quad \beta_{\pm 1} = \pm \frac{y''}{s''},\quad \gamma_{\pm 1} = \pm \frac{z''}{s''},\quad s'' = \sqrt{x''^2 + y''^2 + z''^2}; \tag{11}$$

wir bezeichnen die absolute Wurzel mit s'', weil die Gleichung (6′′) sich jetzt auf: $s''^2 = x''^2 + y''^2 + z''^2$ reducirt. Zugleich folgt aus (6′′) allgemein:

$$s's^{(4)} + 3s''s''' = x'x^{(4)} + 3x''x''' + y'y^{(4)} + 3y''y''' + z'z^{(4)} + 3z''z''' \tag{6′′′}$$

und daher jetzt:

$$s''s''' = x''x''' + y''y''' + z''z'''. \tag{11′}$$

Die Puncte P_{-2}, P, P_2 folgen überdies mit $P_{-2} = P_2$ auf den vereinigten Tangenten $t_{\pm 1}$ rückläufig aufeinander:

Der laufende Punct der Curve ändert an der Stelle P den Sinn seiner Linearfortschreitung.

In den Fällen II, A dürfen die Bedingungen $x'' : y'' : z'' = x''' : y''' : z'''$ nicht erfüllt sein. Als vereinigte Schmiegungsebenen $\Sigma_{\pm 1}$ der Puncte $P_{\pm 1}$ betrachten wir die Verbindungsebenen der Puncte P_1, P_2, P_3 und P_{-1}, P_{-2}, P_{-3} mit der Gleichung:

$$(X-x)(y''z''' - z''y''') + \ldots\ldots + \ldots\ldots = 0.$$

Allerdings müssen wir, um diese Gleichung aus der Entwicklung:

$$\Sigma(t) + \Sigma'(t).\tau + \tfrac{1}{2}\Sigma''(t).\tau^2 + \ldots\ldots = 0,$$

in der mit $x' = 0$, $y' = 0$, $z' = 0$ sowohl $\Sigma(t)$ als $\Sigma'(t)$ verschwinden, zu erhalten, $\tau = \pm\tau_2$ nehmen (vgl. §2). In dieser Auffassung würde also die eben definirte Schmiegungsebene $\Sigma_{\pm 1}$ nicht den Puncten $P_{\pm 1}$, sondern $P_{\pm 2}$ zugehören, was indessen an unseren Betrachtungen sonst nichts wesentliches ändert.

Mit den Schmiegungsebenen $\Sigma_{\pm 1}$ sind die Hauptnormalen $h_{\pm 1}$ ($\perp t_{\pm 1}$ in $\Sigma_{\pm 1}$) und Binormalen $b_{\pm 1}$ ($\perp \Sigma_{\pm 1}$) in $P_{\pm 1}$ bis auf die Pfeilspitzen bestimmt. Diese definiren wir durch die beiden Bedingungen, dass erstens jedes der beiden Coordinatensysteme $\xi_\pm = t_{\pm 1}$, $\eta_\pm = h_{\pm 1}$, $\zeta_\pm = b_{\pm 1}$ positiv orientirt sei und zweitens P_3 und P_{-3} auf der positiven Seite der Ebene $b_1 t_1$ und bezüglich $b_{-1} t_{-1}$ liegen.

Hieraus ergeben sich mit Rücksicht auf (11′) als Richtungscosinus der positiven Hauptnormalen $h_{\pm 1}$:

$$\alpha^{\cdot}_{\pm 1} = \pm \mu \frac{s''x''' - x''s'''}{s''^3}, \quad \beta^{\cdot}_{\pm 1} = \pm \mu \frac{s''y''' - y''s'''}{s''^3}, \quad \gamma^{\cdot}_{\pm 1} = \pm \mu \frac{s''z''' - z''s'''}{s''^3}, \qquad (12)$$

und der positiven Binormalen $b_{\pm 1}$:

$$\alpha^{\cdot\cdot}_{\pm 1} = \mu \frac{y''z''' - z''y'''}{s''^3}, \quad \beta^{\cdot\cdot}_{\pm 1} = \mu \frac{z''x''' - x''z'''}{s''^3}, \quad \gamma^{\cdot\cdot}_{\pm 1} = \mu \frac{x''y''' - y''x'''}{s''^3}, \qquad (12)$$

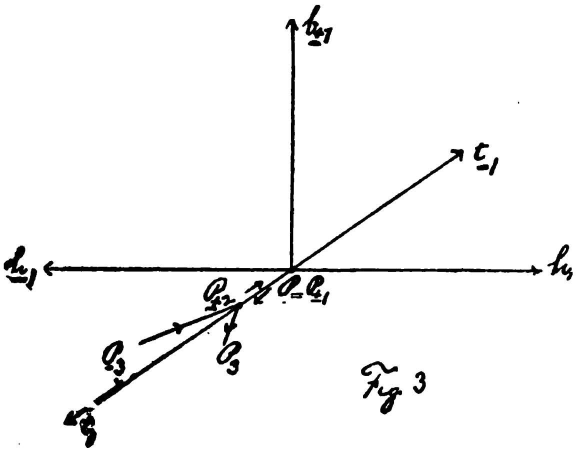
wo die positive Grösse μ durch die Gleichung:

$$\begin{aligned} \frac{s''^3}{\mu} &= \sqrt{(y''z''' - z''y''')^2 + (z''x''' - x''z''')^2 + (x''y''' - y''x''')^2} \\ &= s''\sqrt{x'''^2 + y'''^2 + z'''^2 - s'''^2} \end{aligned}$$

definirt ist. Denn es sind mit den Werthen (11) und (12) die beiden Determinanten $|\alpha_{\pm 1}\beta^{\cdot}_{\pm 1}\gamma^{\cdot\cdot}_{\pm 1}| = +1$ und

$$\eta_{\pm 3} = \alpha^{\cdot}_{\pm 1}(x_{\pm 3} - x) + \beta^{\cdot}_{\pm 1}(y_{\pm 3} - y) + \gamma^{\cdot}_{\pm 1}(z_{\pm 3} - z) = \frac{1}{6}\frac{s''^3}{\mu}\tau_3^3$$

positiv.



Die durch (11) *und* (12) *bestimmten Coordinatensysteme* $t_1 h_1 b_1$ *und* $t_{-1} h_{-1} b_{-1}$ (*vgl. Fig.* 3) *sind allen Fällen II, A charakteristisch.*

In den Fällen II, B sind die Bedingungen $x'' : y'' : z'' = x''' : y''' : z'''$ erfüllt, womit die Puncte $P_{\pm 3}$ in die Tangente $t_{\pm 1}$ zu liegen kommen. Die Formeln (11) behalten ihre Gültigkeit, können aber jetzt noch in anderer Weise dargestellt werden. Es sei nämlich mit einem Proportionalitätsfactor $\varepsilon\varkappa^2$ ($\varepsilon = \pm 1$):

$$x''' = \varepsilon\varkappa^2 x'', \quad y''' = \varepsilon\varkappa^2 y'', \quad z''' = \varepsilon\varkappa^2 z''. \qquad (13)$$

Setzen wir diese Werthe in (11′) ein, so folgt mit Weglassung des Factors s'':

$$s''' = \varepsilon x^2 s'. \tag{13}$$

Wir erhalten so für die Gleichungen (11) die aequivalente Form:

$$\alpha_{\pm 1} = \frac{x'''}{s'''}, \quad \beta_{\pm 1} = \frac{y'''}{s'''}, \quad \gamma_{\pm 1} = \frac{z'''}{s'''}, \quad s''' = \varepsilon\sqrt{x'''^2 + y'''^2 + z'''^2}. \tag{14}$$

Zugleich folgt aus (6‴) allgemein:

$$s's^{(5)} + 4s''s^{(4)} + 3s'''^2 = x'x^{(5)} + 4x''x^{(4)} + 3x'''^2 + \ldots. + \ldots.$$

und daher jetzt mit $x' = 0$, $y' = 0$, $z' = 0$, $s' = 0$ nach (13) und (14):

$$s'''s^{(4)} = x'''x^{(4)} + y'''y^{(4)} + z'''z^{(4)}. \tag{13'}$$

Während nun die Schmiegungsebenen $\Sigma_{\pm 1}$ in den Puncten $P_{\pm 1}$ unbestimmt und die Formeln (13) unbrauchbar werden, betrachten wir als vereinigte Schmiegungsebenen $\Sigma_{\pm 2}$ der Puncte $P_{\pm 2}$ die Verbindungsebenen der Puncte P_2, P_3, P_4 und P_{-2}, P_{-3}, P_{-4} mit der Gleichung:

$$(X - x)(y''z^{(4)} - z''y^{(4)}) + \ldots. + \ldots. = 0$$

$$\text{oder } (X - x)(y'''z^{(4)} - z'''y^{(4)}) + \ldots. + \ldots. = 0$$

(in dem unter II, A angemerkten Sinne den Puncten $P_{\pm 3}$ zugehörig). Mit ihnen sind die Hauptnormalen $h_{\pm 2}$ ($\perp t_{\pm 1}$ in $\Sigma_{\pm 2}$) und Binormalen $b_{\pm 2}$ ($\perp \Sigma_{\pm 2}$) der Puncte $P_{\pm 2}$ bis auf die Pfeilspitzen bestimmt. Diese definiren wir durch die Bedingungen, dass erstens jedes der beiden Coordinatensysteme $\xi_{\pm} = t_{\pm 1}$, $\eta_{\pm} = h_{\pm 2}$, $\zeta_{\pm} = b_{\pm 2}$ positiv orientirt sei und zweitens P_4 und P_{-4} auf der positiven Seite der Ebene $b_2 t_1$ und bezüglich $b_{-2} t_{-1}$ liegt. Hiernach ergeben sich mit Rücksicht auf (13′) als Richtungscosinus der positiven Hauptnormalen $h_{\pm 2}$:

$$\dot{\alpha}_{\pm 2} = \nu\,\frac{s'''x^{(4)} - x'''s^{(4)}}{s'''^2}, \quad \dot{\beta}_{\pm 2} = \nu\,\frac{s'''y^{(4)} - y'''s^{(4)}}{s'''^2}, \quad \dot{\gamma}_{\pm 2} = \nu\,\frac{s'''z^{(4)} - z'''s^{(4)}}{s'''^2}, \tag{15}$$

und der positiven Binormalen $b_{\pm 2}$:

$$\ddot{\alpha}_{\pm 2} = \pm\nu\,\frac{y'''z^{(4)} - z'''y^{(4)}}{s'''^2}, \quad \ddot{\beta}_{\pm 2} = \pm\nu\,\frac{z'''x^{(4)} - x'''z^{(4)}}{s'''^2}, \quad \ddot{\gamma}_{\pm 2} = \pm\nu\,\frac{x'''y^{(4)} - y'''x^{(4)}}{s'''^2}, \tag{15}$$

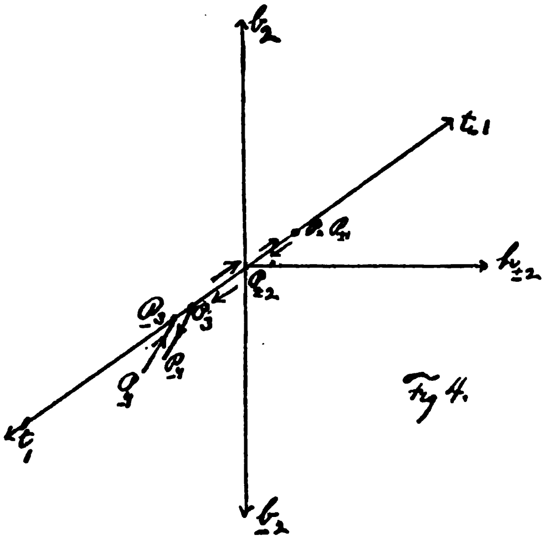
wobei die positive Grösse ν durch die Gleichung:

$$\begin{aligned}\frac{\varepsilon s'''^2}{\nu} &= \sqrt{(y'''z^{(4)} - z'''y^{(4)})^2 + (z'''x^{(4)} - x'''z^{(4)})^2 + (x'''y^{(4)} - y'''x^{(4)})^2}\\ &= \varepsilon s'''\sqrt{x^{(4)2} + y^{(4)2} + z^{(4)2} - s^{(4)2}}\end{aligned}$$

definirt ist. Denn es sind mit diesen Werthen die beiden Determinanten $|\alpha_{\pm 1}\dot\beta_{\pm 2}\ddot\gamma_{\pm 3}| = +1$ und

$$\eta_{\pm 4} = \dot\alpha_{\pm 3}(x_{\pm 4} - x) + \dot\beta_{\pm 3}(y_{\pm 4} - y) + \dot\gamma_{\pm 3}(z_{\pm 4} - z) = \tfrac{1}{24}\frac{s'''^2}{\nu}\tau_4^4$$

positiv. Die Gleichungen (15) sind von dem in (13) eingeführten s unabhängig.



Die durch (14) *und* (15) *bestimmten Coordinatensysteme* $t_1 h_2 b_2$ *und* $t_{-1} h_{-2} b_{-2}$ (*vgl. Fig.* 4) *sind allen Fällen II, B charakteristisch.*

§5. *Der Sinn der Windung.*

In den Figuren 1–4 ist die Curve an der Stelle P nach vorwärts und rückwärts je soweit dargestellt, dass sie bereits mit einem Elemente aus der Tangente heraustritt, also eine Krümmung zeigt oder, anders ausgedrückt, dass der im Sinne des wachsenden Parameters t laufende Punct der Curve vor und nach der Stelle P eine *drehende Bewegung um die zur Binormale parallele Krümmungsaxe* ausführt. Die charakteristischen Coordinatensysteme des §4 sind aber sämmtlich so gewählt, *dass diese drehende Bewegung, von der jeweiligen positiven Binormale aus gesehen, positiv erscheint.*

Nehmen wir jetzt nach vorwärts und rückwärts noch das nächste aus der Schmiegungsebene heraustretende Curvenelement hinzu, so wird der laufende Punct der Curve vor und nach der Stelle P bereits *eine fortschreitende Bewegung* *in der Richtung der Krümmungsaxe* zeigen. Es wird daher, mit Verallgemeinerung der in §1 für die Schraubenlinie gegebenen Definition, *die Windung der Curve* (2) *in dem auf P folgenden Curvenelement positiv oder negativ sein, jenachdem der aus der Schmiegungsebene heraustretende Punct auf deren positiver oder negativer Seite liegt, und die Windung in dem vor P vorausgehenden Curvenelement positiv oder negativ sein, jenachdem der noch nicht in die Schmiegungsebene eingetretene Punct auf deren negativer oder positiver liegt.*

Wir wollen diese allgemeinen Bemerkungen auf die einzelnen Fälle anwenden.

In Bezug auf das Coordinatensystem $\xi = t$, $\eta = h$, $\zeta = b$ der Fälle I, A (vgl. Fig. 1) ist für die unter I, A, a zunächst aus der Schmiegungsebene $\xi\eta$ heraustretenden Puncte $P_{\pm 3}$:

$$\zeta_{\pm 3} = \alpha^{\cdot\cdot}(x_{\pm 3} - x) + \beta^{\cdot\cdot}(y_{\pm 3} - y) + \gamma^{\cdot\cdot}(z_{\pm 3} - z)$$

oder nach (7):

$$\zeta_{\pm 3} = \frac{\rho}{s'^3} \begin{vmatrix} x_{\pm 3} - x & x' & x'' \\ y_{\pm 3} - y & y' & y'' \\ z_{\pm 3} - z & z' & z'' \end{vmatrix} = \pm \tfrac{1}{6} \frac{\rho}{s'^3} \Delta \tau_3^3,$$

wenn wir zur Abkürzung setzen:

$$\Delta = \begin{vmatrix} x' & x'' & x''' \\ y' & y'' & y''' \\ z' & z'' & z''' \end{vmatrix} = |x'y''z'''| = \Delta_{123}\,. \tag{16}$$

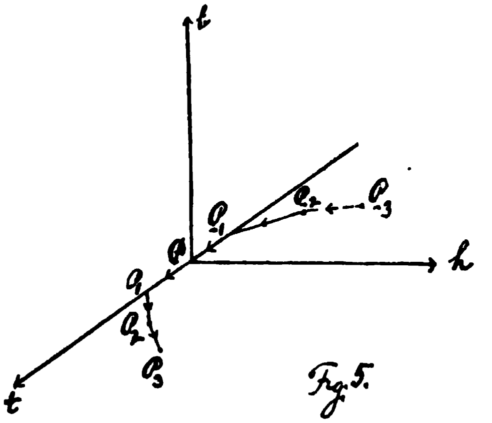


Fig. 5.

Wenn daher $\Delta > 0$ (vgl. Fig. 5, wo das punctirte Element $P_{-3}P_{-2}$ unterhalb der horizontalen *th*-Ebene liegen soll), liegt P_3 auf der positiven, P_{-3} auf der negativen Seite der Schmiegungsebene, wenn $\Delta < 0$, umgekehrt. Es ist also nach der vorausgeschickten Definition die Windung der Curve im Puncte P nach beiden Seiten hin positiv oder negativ, jenachdem $\Delta > 0$ oder < 0 oder wie wir kurz sagen können:

Für I, A, a ist die Windung der Curve im Puncte P positiv oder negativ, jenachdem $\Delta > 0$ *oder* < 0.

In der That ist auch für die in (1) definirte Schraubenlinie die Determinante

$$\Delta = \varepsilon \frac{a^2 b}{2\pi}$$

vom Vorzeichen ε (vgl. §1).

Sind dagegen, wie unter I, A, b, die Puncte $P_{\pm 3}$ in der Schmiegungsebene gelegen, also $\Delta = 0$, so ergiebt sich in Bezug auf das für alle Fälle I, A zu benutzende Coordinatensystem *thb* (vgl. Fig. 1) für die zunächst heraustretenden Puncte $P_{\pm 4}$:

$$\zeta_{\pm 4} = \alpha^{\cdot\cdot}(x_{\pm 4} - x) + \dots + \dots = \tfrac{1}{24}\frac{\rho}{s'^3}\Delta_{124}\tau_4^4,$$

wo Δ_{124} die Determinante:

$$\Delta_{124} = |x' y'' z^{(4)}| = \Delta', \tag{17}$$

den Differentialquotienten von Δ bedeutet. Nach unserer Definition ergiebt sich daher, dass bei positivem Δ_{124} die Windung in dem auf P folgenden Curvenelement positiv, im vorhergehenden negativ ist. Im Puncte P werden wir die *Windung als null* bezeichnen, da die 4 Puncte $PP_1P_2P_3$ in einer Ebene liegen. Wir fassen das Resultat in den Satz zusammen:

Für I, A, b ist die Windung der Curve unmittelbar vor dem Puncte P negativ und unmittelbar nach dem Puncte P positiv oder umgekehrt, jenachdem $\Delta_{124} > 0$ *oder* < 0.

In Bezug auf die Coordinatensysteme $\xi_\pm = t$, $\eta_\pm = h_{\pm 1}$, $\zeta_\pm = b_{\pm 1}$ der Fälle I, B (vgl. Fig. 2) ist für die unter I, B, a zunächst aus der Schmiegungsebene $th_{\pm 1}$ heraustretenden Puncte $P_{\pm 4}$:

$$\zeta_{\pm 4} = \alpha^{\cdot\cdot}_{\pm 1}(x_{\pm 4} - x) + \beta^{\cdot\cdot}_{\pm 1}(y_{\pm 4} - y) + \gamma^{\cdot\cdot}_{\pm 1}(z_{\pm 4} - z),$$

wo wie immer je alle oberen und alle unteren Vorzeichen zusammengehören, oder nach (10):

$$\zeta_{\pm 4} = \pm \tfrac{1}{24}\frac{\lambda}{s''^3}|x'' y''' z^{(4)}|\tau_4^4 = \pm \tfrac{1}{24}\frac{\lambda}{s' s''^3}|x' y''' z^{(4)}|\tau_4^4 = \pm \tfrac{1}{24}\frac{\lambda}{s' s''^3}\Delta_{134}\tau_4^4.$$

Dabei bedeutet:

$$\Delta_{134} = |x' y''' z^{(4)}| = \Delta'' \tag{18}$$

den durch die Voraussetzung $x':y':z'=x'':y'':z''$ reducirten 2. Differentialquotienten $\Delta''=|x'y''x^{(5)}|+|x'y'''z^{(4)}|$ von Δ. Es liegt daher bei positivem Δ_{134} (vgl. Fig. 6) P_4 auf der positiven Seite der Ebene th_1, P_{-4} auf der negativen der Ebene th_{-1} (in Fig. 6 P_4 und P_{-4} beide oberhalb der horizontalen Ebene $th_{\pm 1}$). Die Windung in P selbst ist 0, weil P, P_1, P_2 in gerader Linie liegen und folgt schliesslich:

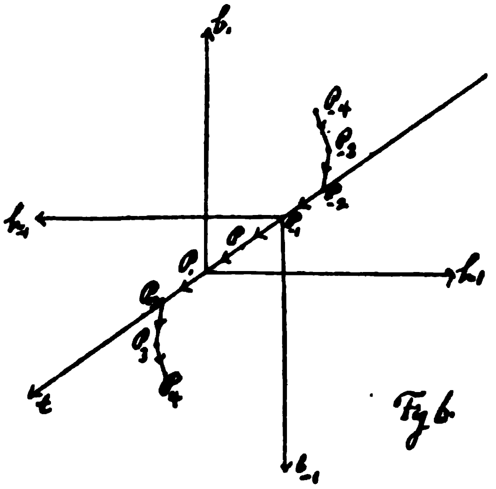


Fig 6

Für I, B, a ist die Windung vor und nach dem Puncte P positiv oder negativ, jenachdem $\Delta_{134}>0$ *oder* <0.

Ist $\Delta_{134}=0$, so reducirt sich die Gleichung:

$$\Delta'''=|x'y''z^{(6)}|+2|x'y'''z^{(5)}|+|x''y'''z^{(4)}|$$

auf:

$$\tfrac{1}{2}\Delta'''=|x'y'''z^{(5)}|=\Delta_{135} \tag{19}$$

und wird für die 3. Coordinate des Punctes P_5 im System th_1b_1 und des Punctes P_{-5} im System $th_{-1}b_{-1}$:

$$\zeta_{\pm 5}=\frac{1}{120}\,\frac{\lambda}{g'g''^2}\,\Delta_{135}\tau_5^5.$$

Es ergiebt sich somit:

Für I, B, b ist die Windung vor dem Puncte P negativ und nach ihm positiv oder umgekehrt, jenachdem $\Delta_{135}>0$ *oder* <0.

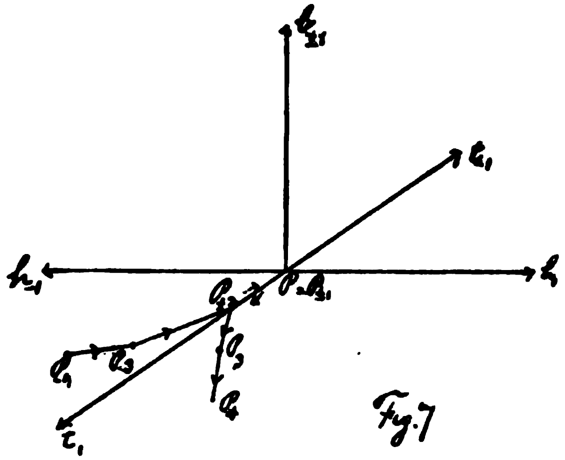
In Bezug auf die Coordinatensysteme der Fälle II, A (vgl. Fig. 3) ist für den Punct $P_{\pm 4}$ nach (12) und (4):

$$\zeta_{\pm 4}=\alpha''_{\pm 1}(x_{\pm 4}-x)+\ldots\ldots+\ldots\ldots=\frac{1}{24}\,\frac{\mu}{g''^2}\,\Delta_{234}\tau_4^4,$$

wo die Determinante:

$$\Delta_{234}=|x''y'''z^{(4)}|=\Delta''' \tag{20}$$

der durch die Bedingungen des Falles II, A, a reducirte 3. Differentialquotient von Δ ist.



Für $\Delta_{234} > 0$ (vgl. Fig. 7) liegt daher P_4 auf der positiven Seite der Ebene $t_1 h_1$ und P_{-4} auf der positiven von $t_{-1} h_{-1}$ (in Fig. 7 beide Puncte oberhalb der horizontalen Ebene $t_{\pm 1} h_{\pm 1}$), also ergiebt sich:

Für II, A, a ist die Windung vor dem Puncte P negativ und nach ihm positiv oder umgekehrt, jenachdem $\Delta_{234} > 0$ *oder* < 0.

Unter der Voraussetzung $\Delta_{234} = 0$ wird entsprechend:

$$\zeta_{\pm 5} = a^{\cdot\cdot}_{\pm 1}(x_{\pm 5} - x) + \ldots + \ldots = \pm \frac{1}{120} \frac{\mu}{s''^3} \Delta_{235} \tau_5^5$$

mit:

$$\Delta_{235} = |x'' y''' z^{(5)}| = \frac{1}{3} \Delta^{(4)}. \tag{21}$$

Für II, A, b ist die Windung vor und nach dem Puncte P positiv oder negativ, jenachdem $\Delta_{235} > 0$ *oder* < 0.

In Bezug auf die Coordinatensysteme der Fälle II, B (vgl. Fig. 4) ist für die Puncte $P_{\pm 5}$ nach (15) und (4):

$$\zeta_{\pm 5} = a^{\cdot\cdot}_{\pm 2}(x_{\pm 5} - x) + \ldots + \ldots = \frac{1}{120} \frac{\nu}{s'' s'''^2} \Delta_{245} \tau_5^5,$$

worin:

$$\Delta_{245} = |x'' y^{(4)} z^{(5)}| = \frac{1}{5} \Delta^{(5)} \tag{22}$$

und $\Delta^{(5)}$ den durch die Bedingungen des Falles II, B, a reducirten 5. Differentialquotienten von Δ bedeutet.

Wenn $\Delta_{245} > 0$ (vgl. Fig. 8) liegt P_5 auf der positiven Seite der Ebene $t_1 h_2$ und P_{-5} der Ebene $t_{-1} h_{-2}$ (in Fig. 8 P_5 oberhalb, P_{-5} unterhalb der horizontalen Ebene $t_{\pm 1} h_{\pm 2}$).

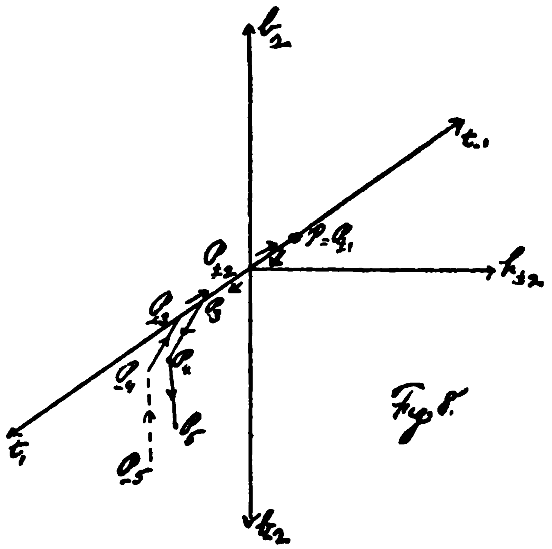


Fig. 8.

Für II, B, a ist die Windung vor dem Puncte P negativ und nach ihm positiv oder umgekehrt, jenachdem $\Delta_{245} > 0$ *oder* < 0.

Unter der Voraussetzung $\Delta_{245} = 0$ wird:

$$\zeta_{\pm 6} = \pm \frac{4}{720} \frac{\nu}{s'' s'''^2} \Delta_{246} \tau_6^6$$

mit:
$$\Delta_{246} = | x'' y^{(4)} z^{(6)} | = \tfrac{1}{16} \Delta^{(6)}. \tag{23}$$

Für II, B, b ist die Windung vor und nach dem Puncte P positiv oder negativ, jenachdem $\Delta_{246} > 0$ *oder* < 0.

Es ist schliesslich zu bemerken, dass diejenigen Differentialquotienten von Δ, welche bezüglich den unter (18) bis (23) angegebenen vorausgehen, in dem betreffenden Falle sämmtlich verschwinden.

§6. *Zusammenfassung der Resultate.*

Wir drücken durch Einschliessen einer Bedingung in eckige Klammern aus, dass dieselbe *nicht erfüllt* ist und *bezeichnen mit zwei nebeneinander gestellten Vorzeichen den Sinn der Windung unmittelbar vor und unmittelbar nach dem Puncte P.* Bei übereinander gestellten Zeichen: $\pm$, $\gtrless$ gehören in jeder Zeile je alle oberen und alle unteren Zeichen zusammen. Mit diesen Festsetzungen giebt die folgende Tabelle zugleich mit den analytischen Kriterien der 8 in §§3, 5 eingeführten Fälle singulärer Puncte jedesmal den Sinn der Windung an.

I. $[x' = 0,\ y' = 0,\ z' = 0]$.
  A. $[x' : y' : z' = x'' : y'' : z'']$.
    a. $|x'\,y''\,z'''| \gtrless 0 : \pm\pm$;
    b. $|x'\,y''\,z'''| = 0,\ \ |x'\,y''\,z^{(4)}| \gtrless 0 : \mp\pm$.
  B. $x' : y' : z' = x'' : y'' : z'',\ \ [x'' : y'' : z'' = x''' : y''' : z''']$.
    a. $|x'\,y'''\,z^{(4)}| \gtrless 0 : \pm\pm$;
    b. $|x'\,y'''\,z^{(4)}| = 0,\ \ |x'\,y'''\,z^{(5)}| \gtrless 0 : \mp\pm$.
II. $x' = 0,\ y' = 0,\ z' = 0,\ \ [x'' = 0,\ y'' = 0,\ z'' = 0]$.
  A. $[x'' : y'' : z'' = x''' : y''' : z''']$.
    a. $|x''\,y'''\,z^{(4)}| \gtrless 0 : \mp\pm$;
    b. $|x''\,y'''\,z^{(4)}| = 0,\ \ |x''\,y'''\,z^{(5)}| \gtrless 0 : \pm\pm$.
  B. $x'' : y'' : z'' = x''' : y''' : z''',\ \ [x''' : y''' : z''' = x^{(4)} : y^{(4)} : z^{(4)}]$.
    a. $|x''\,y^{(4)}\,z^{(5)}| \gtrless 0 : \mp\pm$;
    b. $|x''\,y^{(4)}\,z^{(5)}| = 0,\ \ |x''\,y^{(4)}\,z^{(6)}| \gtrless 0 : \pm\pm$.

Mit Rücksicht auf die Formeln (16)–(23) und die Schlussbemerkung von §5 ist die Determinante, von deren Vorzeichen in jedem Falle der Sinn der Windung abhängt, von einem numerischen Factor abgesehen, *der Werth des niedrigsten nicht verschwindenden Differentialquotienten der Determinante* Δ. Dabei ist der Sinn der Windung vor und nach dem Puncte P derselbe oder nicht derselbe, jenachdem in dem Diagonalgliede jeder Determinante die Summe der oberen Indices (Accente) gerade oder ungerade ist.

Der letztere Umstand stimmt damit überein, dass die Umkehrung der in §2 festgesetzten Durchlaufungsrichtung der Curve mit der Verwandlung des Parameters t der Gleichungen (2) in $-t$ gleichbedeutend ist. Hierbei wechseln diejenigen Determinanten das Vorzeichen, bei welchen die Summe der oberen Indices des Diagonalgliedes ungerade ist. Wenn z. B. im Falle I, A, b an der Stelle P die Determinante $|x'\,y''\,z^{(4)}|$ positiv, also das Doppelzeichen der Windung: $-+$ ist, so ist nach Umkehrung der Durchlaufungsrichtung an derselben Stelle $|x'y''z^{(4)}|$ negativ, das Doppelzeichen der Windung: $+-$. *Bei einer Umkehrung der Durchlaufungsrichtung der Curve bleibt also das Doppelzeichen der Windung in den 4 Fällen I, A, a; I, B, a; II, A, b; II, B, b ungeändert, während es sich in den 4 übrigen Fällen umkehrt.*

Wie sich unsere Tabelle zu der sonst üblichen* Unterscheidung ihrer 8 Fälle verhält, mag noch mit einigen Worten erläutert werden.

Schon in §4 wurde erwähnt, dass der laufende Punct P der Curve in den Fällen I den Sinn seiner Fortschreitungsrichtung nicht ändert, in den Fällen II aber ändert, ein Unterschied, der mit den Symbolen $P = +$ und $P = -$ bezeichnet zu werden pflegt. In gleicher Weise werden die Symbole $t = \pm$ und $\Sigma = \pm$ für die fortschreitende oder zurückkehrende Drehung der Tangente um den laufenden Berührungspunct und der Schmiegungsebene um die laufende Tangente gebraucht.

Die Tangente t im Puncte P ist nach §4 in allen Fällen I durch die Formeln (6) bestimmt. Wir erhalten daher für die Richtungscosinus der nach vorwärts und nach rückwärts benachbarten Tangenten $t_{\pm 1}$ (vgl. §2):

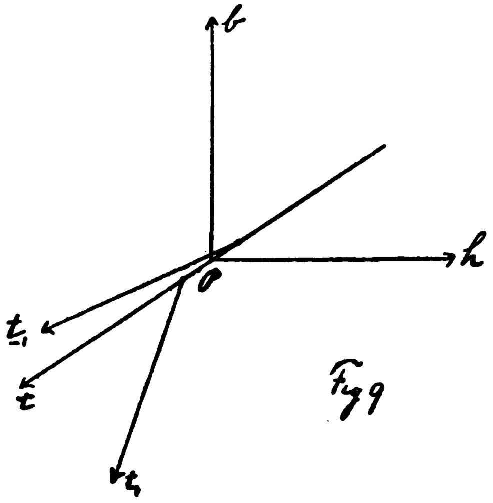
$$\alpha_{\pm 1} = \alpha \pm \alpha' \tau_1, \ldots\ldots, \ldots\ldots$$

Hier ist nach (7):

$$\alpha' = \frac{\varepsilon' \alpha^{\cdot}}{\rho},$$

eine der Frenet'schen Formeln, und ergiebt sich daher für den Winkel der Tangenten $t_{\pm 1}$ gegen die positive Hauptnormale h:

$$\cos(t_{\pm 1}, h) = \alpha_{\pm 1} \alpha^{\cdot} + \ldots\ldots + \ldots\ldots = \pm \frac{\varepsilon'}{\rho} \tau_1.$$



Demnach bildet t_1 mit h einen spitzen, t_{-1} einen stumpfen Winkel (vgl. Fig. 9), die Tangenten t_{-1}, t, t_{+1} folgen, von der positiven Binormale b aus gesehen, im positiven Drehungssinn aufeinander. In Folge der gleichförmigen Definition der Coordinatensysteme des §4 gilt dies aber, wie leicht zu sehen, nicht nur für die Fälle I, A, sondern es dreht sich in allen Fällen beim Durchgang ihres Berührungspunctes durch die Stelle P die Tangente, *von der positiven Binormale aus gesehen, beständig im positiven Sinne.* Jenachdem daher die Binormale selbst an der Stelle P ihre Pfeilspitze beibehält (Fig. 1 und Fig. 3) oder nicht (Fig. 2 und Fig. 4), muss der Sinn der Drehung der Tangente, *von einer und derselben Seite der Schmiegungsebene aus betrachtet, bleiben oder wechseln* unbekümmert darum, dass in den Fällen II an der Stelle P, mit dem Wechsel der Pfeilspitze der Tangente, in der Drehung ein Sprung über 180° stattfindet. In dieser Auffassung ist *für die Fälle I, A und II, A*: $t = +$, *für I, B und II, B*: $t = -$.

---

* Vgl. Ch. Wiener, a. a. O.

Die Binormale b des Punctes P ist nach §4 in allen Fällen I, A durch (7) bestimmt. Wir erhalten daher für die beiderseits benachbarten Binormalen $b_{\pm 1}$ die Richtungscosinus:

$$\alpha^{\cdot\cdot}_{\pm 1} = \alpha^{\cdot\cdot} \pm \alpha^{\cdot\cdot\prime}\tau_1, \ldots\ldots, \ldots\ldots$$

Hierin ist:

$$\alpha^{\cdot\cdot\prime} = -\frac{\rho^2\Delta}{s'^3}\alpha^{\cdot},$$

eine der Frenet'schen Formeln, die sich aus (7) durch Differentiation und geeignete Reduction ergiebt. Für die Winkel der Binormalen $b_{\pm 1}$ gegen die Hauptnormale h ergiebt sich somit:

$$\cos(b_{\pm 1}, h) = \alpha^{\cdot\cdot}_{\pm 1}\alpha^{\cdot} + \ldots\ldots + \ldots\ldots = \mp\frac{\rho^2\Delta}{s'^3}\tau_1.$$

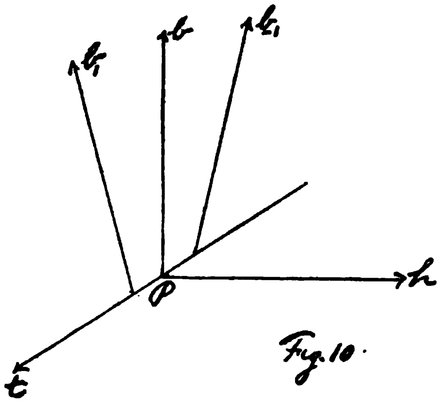


Fig. 10.

Bei positivem Δ (vgl. Fig. 10) bildet also b_1 einen stumpfen, b_{-1} einen spitzen Winkel mit h, sodass von der positiven Tangente aus gesehen, die Binormalen

b_{-1}, b, b_1 im positiven Drehungssinn auf einander folgen, bei negativem Δ umgekehrt. Bezeichnen wir also durch zwei nebeneinander gestellte Vorzeichen den Sinn der Drehung der positiven Binormale unmittelbar vor und unmittelbar nach dem Puncte P, gesehen von der positiven Tangente aus, so stellt sich unser Resultat durch die Formel dar: $\Delta \gtrless 0 : \pm\pm$, übereinstimmend mit dem Doppelzeichen der Windung im Falle I, A, a. In Folge der gleichförmigen Definition der charakteristischen Coordinatensysteme des §4 und des Sinnes der Windung in §5 gilt diese Beziehung aber nicht nur im Falle I, A, a, sondern auch in allen übrigen Fällen. Es ergiebt sich damit eine zweite Bedeutung der vorhin aufgestellten Tabelle. Sie giebt, mit dem Sinn der Windung in den 8 Fällen, zugleich *den Sinn der Drehung der positiven Binormale unmittelbar vor und unmittelbar nach dem Puncte P, wie er von der positiven Tangente aus erscheint*, unbekümmert um einen in den Fällen I, B und II, B stattfindenden Sprung über 180° im Puncte P. Jenachdem daher die positive Tangente selbst in P ihre Pfeilspitze beibehält (Fälle I) oder wechselt (Fälle II), bedeutet das Symbol: $\pm\pm$ oder das Symbol $\mp\pm$ eine Erhaltung des Drehungssinnes der Binormale b oder der Schmiegungsebene Σ *für einen unveränderlichen Standpunct.* In dieser Auffassung ist für die 4 Fälle a: $\Sigma = +$, für die 4 Fälle b: $\Sigma = -$.

Nach diesen Bemerkungen können die 8 Fälle auch in der bekannten Weise bezeichnet werden:

	I				II			
	A		B		A		B	
	a	b	a	b	a	b	a	b
P	+	+	+	+	−	−	−	−
t	+	+	−	−	+	+	−	−
Σ	+	−	+	−	+	−	+	−

Während hiermit aber nur die Frage beantwortet wird, ob der Sinn der Bewegung von Punct, Tangente und Schmiegungsebene sich umkehrt oder nicht umkehrt, enthält unsere Darstellung die vollständige analytische Bestimmung des Sinnes aller dieser Bewegungen in Bezug auf das zu Grunde liegende Coordinatensystem, wie sie namentlich für die Untersuchung der Bewegung eines Punctes im Raume von Vortheil ist.

ROSTOCK, *März* 1895.

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